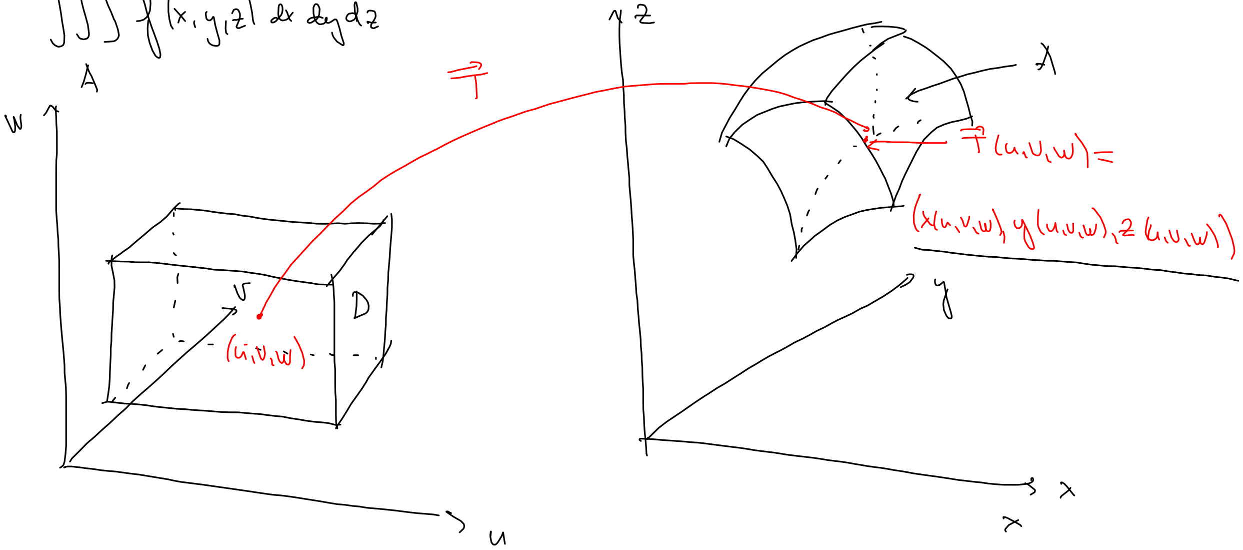
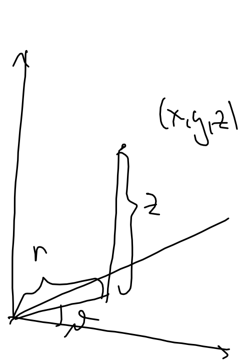
Skifte av variabel i trippelintegraler

$$\iiint_A f(x,y,z)\,dx\,dy\,dz$$



$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_D f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

der
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$



Sylinderkoordinater: Nye koordinater: r, θ, z

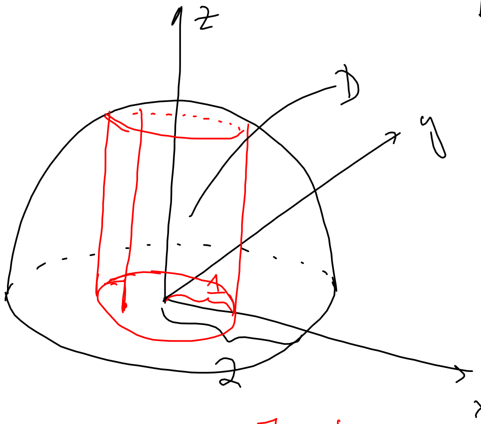
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Jacobi-determinant:
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta & 0 \\ 0 & 1 \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = \underline{r}$$

Eksempel Regn ut $I = \iiint_A x^2 z \, dx\, dy\, dz$ der A er området som ligger over xy-planet og inni både kulen om origo med radius 2 og den stående sylinderen med radius 1 om origo.



Skifter til sylinderkoordinater:

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

Jacobi-determinant

$$I = \iiint_D (r\cos\theta)^2 z \cdot r \, dr\, d\theta\, dz = \int_0^1 \left[\int_0^{2\pi} \left[\int_0^{\sqrt{4-r^2}} r^3\cos^2\theta\, z \, dz\right] d\theta\right] dr$$

Mellomregning for øvre grense for z:
$x^2+y^2+z^2 = 4 \Rightarrow z = \sqrt{4-x^2-y^2}$
$= \sqrt{4-r^2}$

$$= \int_0^1 \left[\int_0^{2\pi} r^3\cos^2\theta \left[\frac{1}{2}z^2\right]_{z=0}^{z=\sqrt{4-r^2}} d\theta\right] dr$$

$$= \frac{1}{2}\int_0^1 \left[\int_0^{2\pi} r^3\cos^2\theta\,(4-r^2)\, d\theta\right] dr$$

$$= \frac{1}{2}\int_0^1 r^3(4-r^2)\underbrace{\left[\int_0^{2\pi}\cos^2\theta\, d\theta\right]}_{\pi} dr$$

Mellomregning: $\int_0^{2\pi} \cos^2\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\Downarrow$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{1}{2}\int_0^{2\pi}(\cos 2\theta + 1)\, d\theta$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[2\pi\right] = \pi$

$$= \frac{\pi}{2}\int_0^1 (4r^3 - r^5)\, dr = \frac{\pi}{2}\left[r^4 - \frac{r^6}{6}\right]_0^1$$

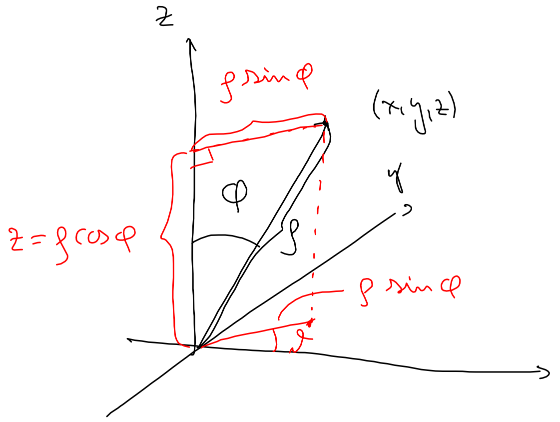
$$= \frac{\pi}{2}\left[1 - \frac{1}{6}\right] = \frac{\pi}{2}\cdot\frac{5}{6} = \underline{\underline{\frac{5\pi}{12}}}$$

HUSK: Undervisningsbytte onsdag/torsdag:

Onsdag: Forelesning

Torsdag: Plenumsregning.

Kulekoordinater



$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

Jacobi-determinanten:

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cancel{\sin\varphi\cos\theta} & \cancel{\rho\cos\varphi\cos\theta} & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \sin\varphi\cos\theta\, \rho^2 \sin^2\varphi \cos\theta + \rho\cos\varphi\cos\theta\, \rho\cos\varphi\sin\varphi\cos\theta$$
$$- \rho\sin\varphi\sin\theta\left(-\rho\sin^2\varphi\sin\theta - \rho\cos^2\varphi\sin\theta\right)$$

$$= \rho^2\sin^3\varphi\cos^2\theta + \rho^2\cos^2\varphi\sin\varphi\cos^2\theta$$
$$+ \rho^2\sin^3\varphi\sin^2\theta + \rho^2\cos^2\varphi\sin\varphi\sin^2\theta$$

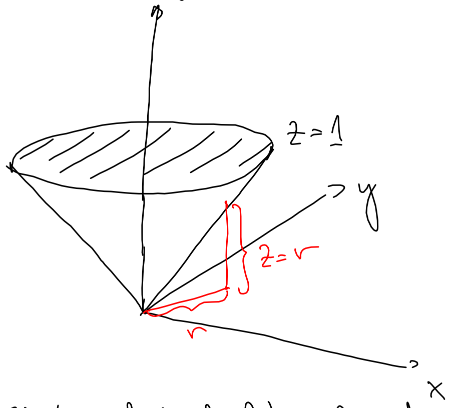
$$= \rho^2\sin^3\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + \rho^2\cos^2\varphi\sin\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$

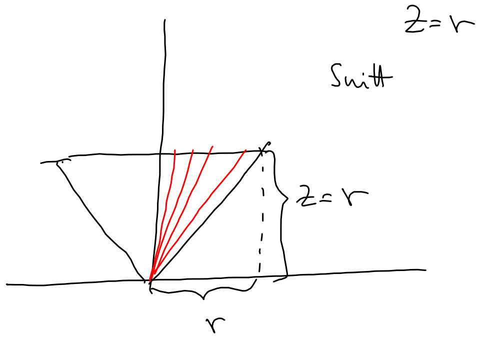
$$= \rho^2\sin^3\varphi + \rho^2\cos^2\varphi\sin\varphi = \rho^2\sin\varphi\,[\underbrace{\sin^2\varphi + \cos^2\varphi}_{1}] = \underline{\rho^2\sin\varphi}$$

Oppsummering: Når vi skifter til kulekoordinater, blir Jacobi-determinanten

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi$$

Eksempel: Regn ut $I = \iiint_A z \, dx\,dy\,dz$ der A er området inni kjeglen $z = \sqrt{x^2+y^2}$ og under planet $z = 1$





Skifter til kulekoordinater:

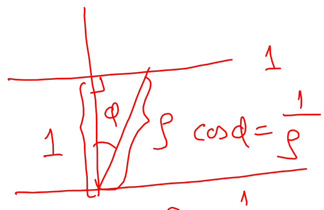
$$I = \iiint_A z\,dx\,dy\,dz = \iiint_D \rho\cos\varphi \underbrace{\rho^2\sin\varphi}_{\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}} \, d\rho\, d\varphi\, d\theta$$

A beskrevet i kulekoordinater

$0 \le \varphi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

Regner på ρ:



$\cos\varphi = \frac{1}{\rho}$

$\rho = \frac{1}{\cos\varphi}$

$0 \le \rho \le \frac{1}{\cos\varphi}$

$$= \int_0^{\frac{\pi}{4}} \left[\int_0^{2\pi} \left[\int_0^{\frac{1}{\cos\varphi}} \rho^3 \cos\varphi \sin\varphi \, d\rho \right] d\theta \right] d\varphi$$

$$= \int_0^{\pi/4} \left[\int_0^{2\pi} \cos\varphi\sin\varphi \int_0^{\frac{1}{\cos\varphi}} \rho^3 \, d\rho \right] d\theta \Big] d\varphi$$

$$= \int_0^{\pi/4} \left[\int_0^{2\pi} \cos\varphi \sin\varphi \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=\frac{1}{\cos\varphi}} d\theta \right] d\varphi =$$

$$= \frac{1}{4} \int_0^{\pi/4} \left[\int_0^{2\pi} \cos\varphi \sin\varphi \frac{1}{\cos^4\varphi} \, d\theta \right] d\varphi$$

$$= \frac{1}{4} \int_0^{\pi/4} \left[\int_0^{2\pi} \frac{\sin\varphi}{\cos^3\varphi} \, d\theta \right] d\varphi = \frac{2\pi}{4} \int_0^{\pi/4} \frac{\sin\varphi}{\cos^3\varphi} \, d\varphi$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$\sin\varphi \, d\varphi = -du$

$$= \frac{\pi}{2} \int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u^3} (-du) = \frac{\pi}{2} \int_{\frac{\sqrt{2}}{2}}^1 u^{-3} \, du = \ldots\ldots$$

$\varphi = 0$: $u = \cos 0 = 1$

$\varphi = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Gjør'e sjøl!

Anvendelser av trippelintegraler

Volum: $\text{Volum}(R) = \iiint_R 1 \, dx\, dy\, dz$

Masse: **Masse**: $\iiint_R \rho(x,y,z) \, dx\, dy\, dz = M$

$\rho(x,y,z)$ tettheten i punktet x,y,z

Massemiddelpunkt: $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{\iiint x \rho(x,y,z) \, dx\, dy\, dz}{M}$$

$$\bar{y} = \frac{\iiint y \rho(x,y,z) \, dx\, dy\, dz}{M}$$

$$\bar{z} = \frac{\iiint z \rho(x,y,z) \, dx\, dy\, dz}{M}$$