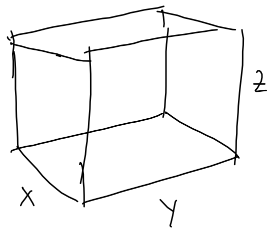
<u>Eksempel</u>: Kasse uten lokk: Totale lengde stenger 56 m

Maksimere overflaten: F



$$4x + 4y + 4z = 56$$

$$z = 14 - x - y$$

$$F = xy + 2xz + 2yz$$

$$= xy + 2x(14-x-y) + 2y(14-x-y) \qquad x>0,\ y>0$$

$$= 28x + 28y - 3xy - 2x^2 - 2y^2 \qquad x+y \le 14$$

<u>Deriverer</u>:

$$\frac{\partial F}{\partial x} = 28 - 3y - 4x, \quad \frac{\partial F}{\partial y} = 28 - 3x - 4y$$

Får:

$$\begin{aligned} 4x + 3y &= 28 \quad \Big|\cdot 4 \\ 3x + 4y &= 28 \quad \Big|\cdot(-3) \end{aligned} \Rightarrow \begin{aligned} 16x + 12y &= 4\cdot 28 \\ -9x - 12y &= (-3)\cdot 28 \end{aligned}$$

$$7x = 28 \Rightarrow x = 4 \quad y = 4$$

$$z = 14 - x - y = 6$$

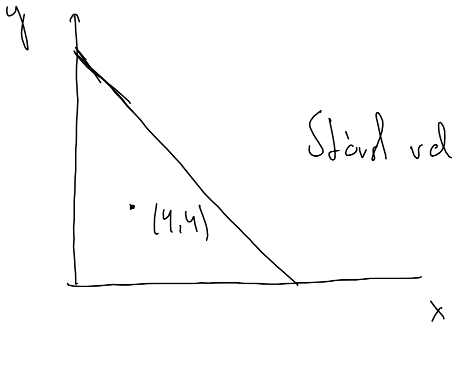
<u>Mulig maks</u>: $(4,4,6)$

<u>Annenderivertesten</u>. $A = \frac{\partial^2 F}{\partial x^2}(4,4) = -4$, $B = \frac{\partial^2 F}{\partial x \partial y}(4,4) = -3$

$$C = \frac{\partial^2 F}{\partial y^2}(4,4) = -4$$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix} = (-4)(-4) - (-3)(-3) = 16 - 9 = 7 > 0$$

Ser at $D > 0$, $A < 0$, altså er $(4,4)$ et lokalt maksimum.



$$x + y \le 14$$

Største volum får vi for $x=4, y=4, z=6$

$$F(4,4,6) = 4\cdot 4 + 2\cdot 4\cdot 6 + 2\cdot 4 \cdot 6$$

$$= 16 + 48 + 48 = \underline{112\,m^2}$$

Optimering under bibetingelser

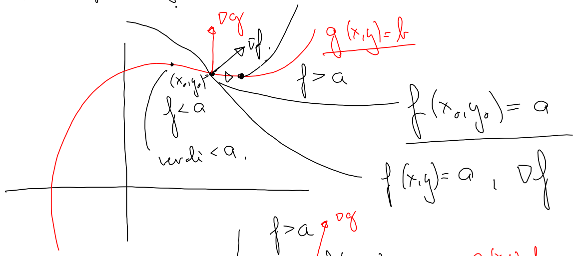
Hittil: Maks/min. $f(\vec{x})$ når det ikke er begrensninger på $\vec{x}$.

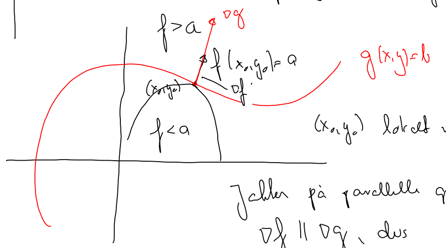
Nå: — " — $f(\vec{x})$ når vi har en begrensning av typen

$$\underbrace{g(\vec{x})=b}_{\text{bibetingelse.}}$$

Konkretiserer til 2-dim: Ønsker å maksimere $f(x,y)$ under bibetingelsen $g(x,y)=b$.

verdi $= a$





(x_0,y_0) lokalt min.

Gjelder på parallelle gradienter $\nabla f \parallel \nabla g$, dvs

$$\nabla f = \lambda \nabla g$$

λ: Lagrange-multiplikator.

Teorem. Anta at $f,g:\mathbb{R}^m \to \mathbb{R}$ er to funksjoner med kontinuerlige partiellderiverte. Anta at $\vec{x}$ er et lokalt maksimum eller minimum for f på mengden

$$A=\{\vec{x}\in\mathbb{R}^m \mid g(\vec{x})=b\}.$$

Da er enten $\nabla g(\vec{x})=\vec{0}$ eller det finnes et tall λ slik at $\nabla f(\vec{x})=\lambda\nabla g(\vec{x})$

I detalj: $f(x_1,x_2,\dots,x_m)$, $g(x_1,\dots,x_m)$

$$\nabla f(\vec{x})=\begin{pmatrix}\frac{\partial f}{\partial x_1}(\vec{x})\\ \frac{\partial f}{\partial x_2}(\vec{x})\\ \vdots\\ \frac{\partial f}{\partial x_m}(\vec{x})\end{pmatrix},\quad \nabla g(\vec{x})=\begin{pmatrix}\frac{\partial g}{\partial x_1}(\vec{x})\\ \vdots\\ \frac{\partial g}{\partial x_m}(\vec{x})\end{pmatrix}$$

Vi får

m ligninger med m+1 ukjente $x_1,x_2,\dots,x_m,\lambda$

$$\begin{cases}\frac{\partial f}{\partial x_1}(\vec{x})=\lambda\cdot\frac{\partial g}{\partial x_1}(\vec{x}) & \vec{x}=(x_1,\dots,x_m)\\ \frac{\partial f}{\partial x_2}(\vec{x})=\lambda\frac{\partial g}{\partial x_2}(\vec{x})\\ \vdots\\ \frac{\partial f}{\partial x_m}(\vec{x})=\lambda\frac{\partial g}{\partial x_m}(\vec{x})\end{cases}$$

Husk: Én ligning til! $\{\; g(\vec{x})=b$

VB avd. 4, kl $16^{\underline{15}}$

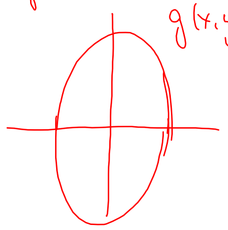
Eksempel: Finn maks og min. til

$$f(x,y) = xy$$

under bibetingelsen

$$\underbrace{9x^2 + y^2 = 18}, \quad g(x,y) = 9x^2 + y^2, \; b = 18$$

Geometrisk



Ser etter punkter slik at $\nabla f = \lambda \nabla g$.

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\nabla g(x,y) = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 18x \\ 2y \end{pmatrix}$$

Setter inn i $\nabla f = \lambda \nabla g$:

$$\left.\begin{aligned} y &= 18\lambda x \\ x &= 2\lambda y \\ 9x^2 + y^2 &= 18 \end{aligned}\right\}$$

Har delt til i alle den første ligningen med den andre: Problem: kan x, y eller λ være lik 0. Ser at dette er umulig!

Deler: $\frac{y}{x} = \frac{18\cancel{\lambda} x}{2\cancel{\lambda} y} = \frac{9x}{y}$

$$\Downarrow$$

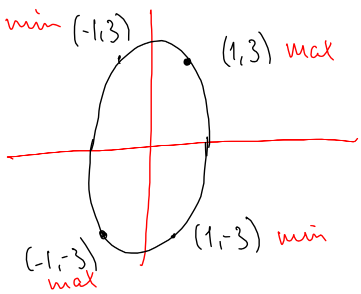
$$y^2 = 9x^2$$

Setter inn i den tredje ligningen:

$$9x^2 + 9x^2 = 18 \Rightarrow x^2 = 1 \quad \begin{cases} x = 1: \; y^2 = 9 \Rightarrow y = \pm 3 \\ x = -1: \; y^2 = 9 \Rightarrow y = \pm 3 \end{cases}$$

Løsninger: $(1,3), (1,-3), (-1,3), (-1,-3)$.

$$f(x,y) = xy \qquad \begin{aligned} &f(1,3) = 1 \cdot 3 = 3, \quad f(1,-3) = 1 \cdot (-3) = -3 \\ &f(-1,3) = (-1) \cdot 3 = -3, \quad f(-1,-3) = (-1)(-3) = 3 \end{aligned}$$



Flere bibetingelser:

Max/min til $f(x_1,..,x_n)$ under bibetingelsene

$$g_1(x_1,..,x_n)=b_1$$
$$g_2(x_1,..,x_n)=b_2$$
$$\vdots$$
$$g_k(x_1,..x_n)=b_k$$

Teorem: Anta at $f, g : \mathbb{R}^n \to \mathbb{R}$ har kontinuerlige partiellderiverte. Dersom $\bar{x}$ er et lokalt maks. eller minimum for f på mengden

$$A = \{\vec{x} \in \mathbb{R}^n \mid g_1(\bar{x}) = b_1, g_2(\bar{x}) = b_2, .. \text{ og } g_k(\bar{x}) = b_k\},$$

så er enten $\nabla g_1(\bar{x}), \nabla g_2(\bar{x}), \ldots, \nabla g_k(\bar{x})$ lineært avhengige eller det finnes konstanter $\lambda_1, \lambda_2, .., \lambda_k$ slik at

$$\nabla f(\bar{x}) = \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) + \cdots + \lambda_k \nabla g_k(\bar{x})$$

Eksempel: Finn minimum til funksjonen

$$f(x,y,z) = x^2 - 2x + 2y^2 + z^2 + z$$

under bibetingelsene $\underbrace{x+y+z}_{g_1(x,y,z)} = 1$, $\underbrace{2x-y-z}_{g_2(x,y,z)} = 5$

Sjekk punkter der $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

$$\nabla f = \begin{pmatrix} 2x-2 \\ 4y \\ 2z+1 \end{pmatrix}, \quad \underbrace{\nabla g_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \nabla g_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}_{\text{lin. uavh.}}$$

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$:

3 lign. med 5 ukjente $x, y, z, \lambda_1, \lambda_2$

$$\begin{cases} 2x-2 = \lambda_1 + 2\lambda_2 \\ 4y = \lambda_1 - \lambda_2 \\ 2z+1 = \lambda_1 - \lambda_2 \end{cases}$$

$\Rightarrow 4y = 2z+1 \Rightarrow 2z = 4y-1$

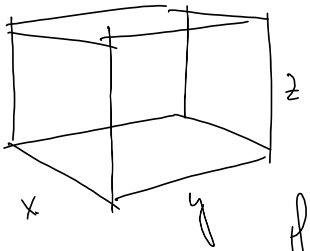
$z = 2y - \frac{1}{2}$

$$\left.\begin{aligned} x+y+z=1 \\ 2x-y-z=5 \end{aligned}\right\} \Leftarrow z = 2y - \tfrac{1}{2}$$

$x=2, \; y=-\frac{1}{6}, \; z=-\frac{5}{6}$

Uoppstilte maks/min-problemer:

Eksempel:



Kasse uten lokk: Rør 56 m

Maksimerer:

$$f(x,y,z) = xy + 2xz + 2yz$$

$$4x + 4y + 4z = 56$$

$$\underbrace{x+y+z}_{g(x,y,z)} = 14$$ bibetingelse

Bruker Lagrange:

$$\nabla f = \begin{pmatrix} y+2z \\ x+2z \\ 2x+2y \end{pmatrix}, \quad \nabla g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\nabla f = \lambda \nabla g$:

$$\left.\begin{aligned} y+2z &= \lambda \\ x+2z &= \lambda \end{aligned}\right\} \Rightarrow x = y$$

$$\left.\begin{aligned} x+2z &= \lambda \\ 2x+2y &= \lambda \end{aligned}\right\} \Rightarrow x+2z = 2x+2y \Rightarrow x+2y-2z=0$$

$$x+y+z=14$$

Har:

$$x = y$$

$$\left.\begin{aligned} x+2y-2z &= 0 \\ x+y+z &= 14 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3x-2z &= 0 \\ 2x+z &= 14 \end{aligned}\right\} \Rightarrow \begin{aligned} 3x-2z &= 0 \\ 4x+2z &= 28 \end{aligned}$$

$$7x = 28 \Rightarrow x = 4$$

$$y = 4$$

$$z = 6$$