Plenumsregning: Torsdag 19/2, Mandag 23/2

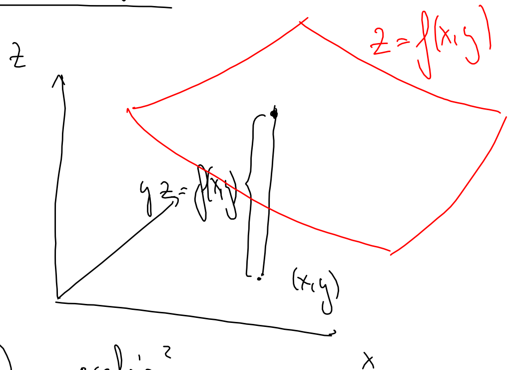
Forelesninger: Onsdag 18/2, Onsdag 25/2

Grafisk fremstilling av skalarfelt (3.7)

Skalarfelt: $f: \mathbb{R}^n \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$, $f: \mathbb{R}^3 \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$, $z = f(x,y)$



Hvordan tegner vi $z=f(x,y)$ i praksis?

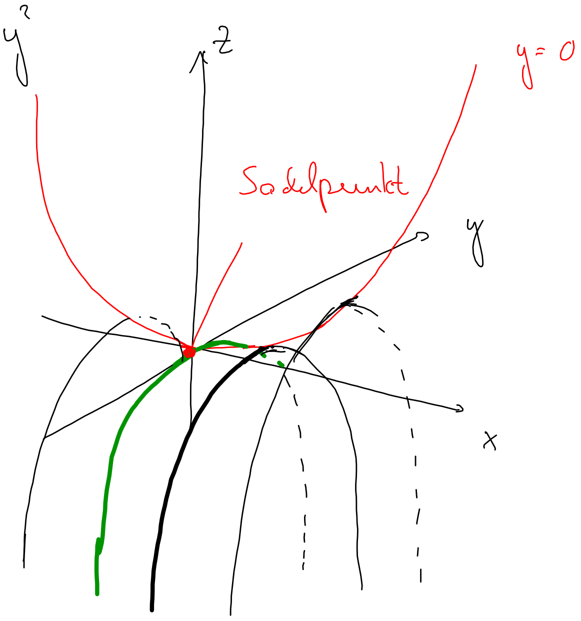
Eksempel: $z = f(x,y) = x^2 - y^2$

$y=0$: $z = f(x,0) = x^2$

$x=0$: $z = f(0,y) = -y^2$

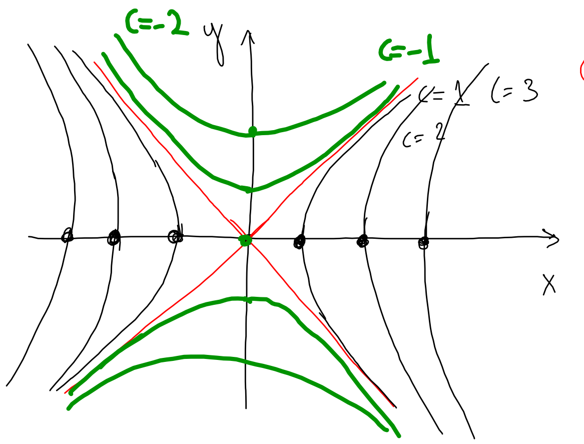
$x=1$ $z = f(1,y) = 1-y^2$

$x=2$ $z = f(2,y) = 4-y^2$



Nivåkurver: $c \in \mathbb{R}$

$N_c = \{(x,y) : f(x,y) = c\}$ $c \geq 0$: $x^2 - y^2 = c \Rightarrow \frac{x^2}{(\sqrt{c})^2} - \frac{y^2}{(\sqrt{c})^2} = 1$

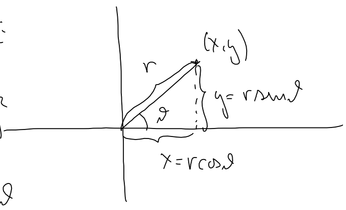


$c < 0$ $x^2 - y^2 = c$

$\frac{y^2}{-c} - \frac{x^2}{-c} = 1$

$\frac{y^2}{(\sqrt{-c})^2} - \frac{x^2}{(\sqrt{-c})^2} = 1$

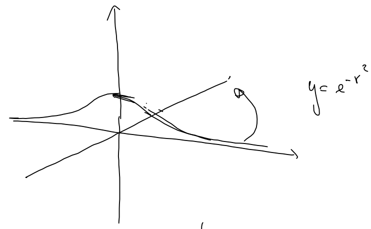
Polarkoordinater:



Eks: $f(x,y)=e^{-x^2-y^2}$

$= e^{-r^2\cos^2\theta - r^2\sin^2\theta}$

$= e^{-r^2(\cos^2\theta+\sin^2\theta)} = e^{-r^2}$



Funksjoner av tre variable: $u=f(x,y,z)$ | 4-dimensjonal.

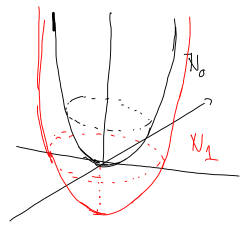
Nivåflater: $c\in\mathbb{R}$, $N_c=\{(x,y,z): f(x,y,z)=c\}$

Eksempel: $f(x,y,z)=x^2+y^2-z$

$c=x^2+y^2-z \Rightarrow \boxed{z=x^2+y^2-c}$

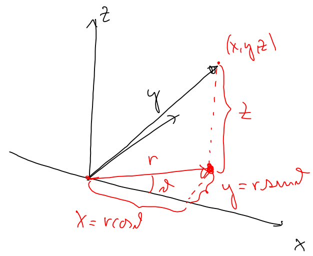
$c=0$: $z=x^2+y^2$

$c=1$: $z=x^2+y^2-1$



Sylinderkoordinater

$f(x,y,z)=x^2+y^2-z=\underline{r^2-z}$

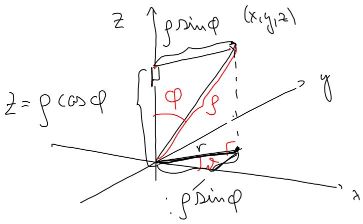


Kulekoordinater,

(ρ,φ,θ), $\rho\ge 0$

$0\le\varphi\le\pi$

$0\le\theta\le 2\pi$



$x=\rho\sin\varphi\cos\theta$

$y=\rho\sin\varphi\sin\theta$

$$\boxed{\begin{aligned} x&=\rho\sin\varphi\cos\theta\\ y&=\rho\sin\varphi\sin\theta\\ z&=\rho\cos\varphi. \end{aligned}}$$

Eksempel: $f(x,y,z)=x^2+y^2-z^2$

$=(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2-(\rho\cos\varphi)^2$

$=\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta-\rho^2\cos^2\varphi$

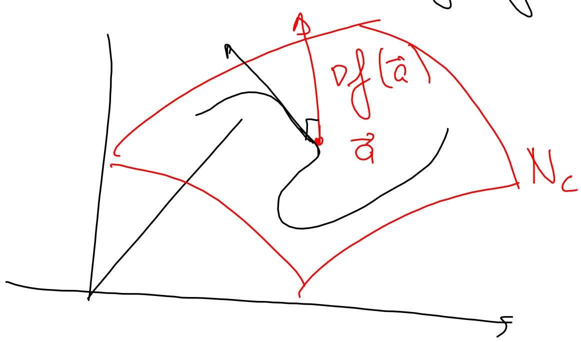
$=\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)-\rho^2\cos^2\varphi$

$=\rho^2\sin^2\varphi-\rho^2\cos^2\varphi=-\rho^2\underbrace{(\cos^2\varphi-\sin^2\varphi)}_{\cos 2\varphi}=\underline{-\rho^2\cos 2\varphi}$

Anta at $f: \mathbb{R}^n \to \mathbb{R}$ er en funksjon av n-variable. Hvis $c \in \mathbb{R}$, så kalles

$$N_c = \{(x_1, x_2, .., x_n) : f(x_1, x_2, .., x_n) = c\}$$

kalles en nivåflate for f.



$f(\vec{a}) = c$

Påstår: $\nabla f(\vec{a})$ er en normal til nivåflaten N_c

Setning: Anta at f er en deriverbar funksjon og at $f(\vec{a}) = c$. Hvis $\vec{r}$ er en deriverbar kurve på flaten N_c slik at $\vec{r}(t) = \vec{a}$, så er

$$\nabla f(\vec{a}) \cdot \vec{r}\,'(t) = 0.$$

Bevis: Siden $\vec{r}$ ligger på N_c, så er $f(\vec{r}(s)) = c$ for alle s. Deriverer på begge sider

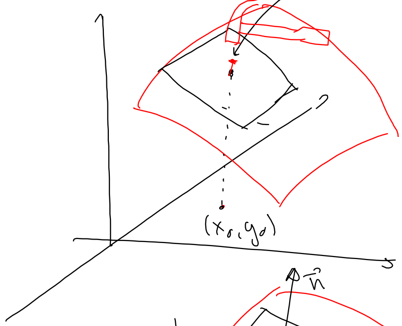
$$\nabla f(\vec{r}(s)) \cdot \vec{r}\,'(s) = 0 \quad \text{for alle } s.$$

Setter inn $s = t$:

$$\nabla f(\vec{a}) \cdot \vec{r}\,'(t) = 0.$$

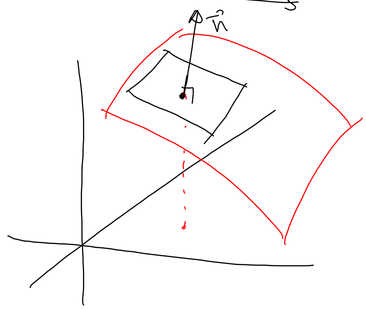
Tangentplan

$z = f(x,y)$



$z = f(x,y)$

Tangentplan



Planet gjennom (x_0,y_0,z_0) med samme normalvektor som flaten selv.

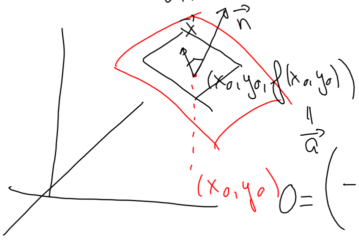
Hvordan finner vi en normalvektor til flaten $z = f(x,y)$? Sabla juks: Gjør om dette til en nivåflate.

$$g(x,y,z) = z - f(x,y) \; : \; N_0 = \{(x,y,z) : \underbrace{z - f(x,y) = 0}_{z = f(x,y)}\}$$

Normalvektoren til flaten er dermed gradienten til g

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$\vec{n} = -\frac{\partial f}{\partial x}(x_0,y_0)\vec{i} - \frac{\partial f}{\partial y}(x_0,y_0)\vec{j} + \vec{k}$$



Ligningen til planet:

$$\vec{n} \cdot (\vec{x} - \vec{a}) = 0$$

$$0 = \left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right) \cdot (x - x_0, y - y_0, z - f(x_0,y_0))$$

$$= -\frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + z - f(x_0,y_0)$$

dvs

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

Tangentplanet til $z = f(x,y)$ i punktet $(x_0,y_0,f(x_0,y_0))$ er

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

og normalvektoren i dette punktet er:

$$\vec{n} = -\frac{\partial f}{\partial x}(x_0,y_0)\vec{i} - \frac{\partial f}{\partial y}(x_0,y_0)\vec{j} + \vec{k}$$