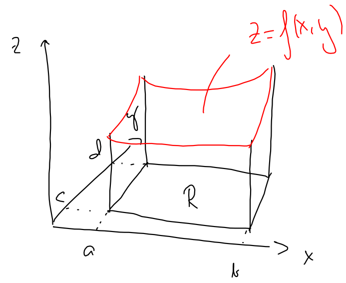
Dobbeltintegraler



$z = f(x,y)$

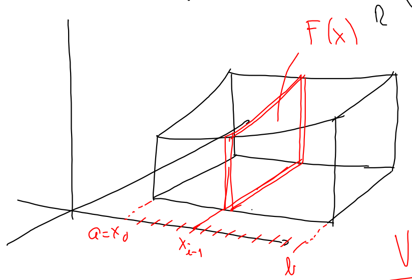
f integrerbar

$$\iint_R f(x,y)\,dx\,dy = \iint f(x,y)\,dx\,dy$$

$$\iint_R f(x,y)\,dx\,dy$$

Teorem: $f : R \to \mathbb{R}$ er kontinuerlig, så er f integrerbar.

Hvordan regner vi ut $\iint_R f(x,y)\,dx\,dy$?



$$F(x) = \int_c^d f(x,y)\,dy$$

$$V_i \approx F(x_{i-1})(x_i - x_{i-1})$$

Totalt volum:

$$V \approx \sum V_i = \underbrace{\sum_{i=1}^{n} F(x_{i-1})(x_i - x_{i-1})}_{\text{Riemann-sum}}$$

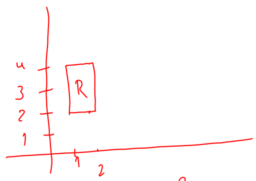
$$\to \int_a^b F(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

iterert integral

Teorem: Anta at $f : R \to \mathbb{R}$ er kontinuerlig. Da er

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Eksempel: Regn ut $I = \iint_R x^2 y\,dx\,dy$ $\quad R = [1,2] \times [2,4]$



$$I = \int_1^2 \left[\int_2^4 x^2 y\,dy\right] dx = \int_1^2 \left[\tfrac{1}{2} x^2 y^2\right]_{y=2}^{y=4} dx$$

$$= \int_1^2 \Big[\overbrace{\tfrac{1}{2} x^2 \cdot 16}^{8x^2} - \overbrace{\tfrac{1}{2} x^2 \cdot 4}^{2x^2}\Big] dx = \int_1^2 6x^2\,dx = \left[6 \cdot \frac{x^3}{3}\right]_1^2 = \left[2x^3\right]_1^2 = 16 - 2 = \underline{\underline{14}}$$

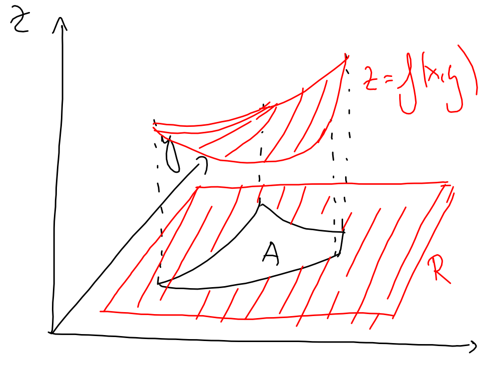
Alternativt:

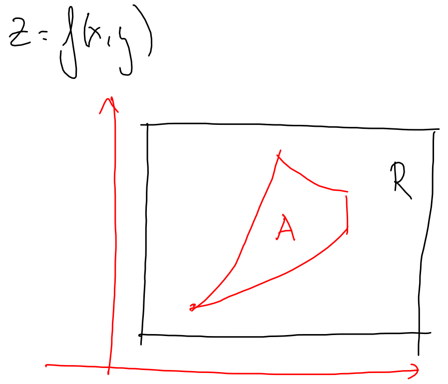
$$I = \int_2^4 \left[\int_1^2 x^2 y\,dx\right] dy = \int_2^4 \left[\frac{x^3}{3} y\right]_{x=1}^{x=2} dy$$

$$= \int_2^4 \left[\frac{8}{3} y - \frac{1}{3} y\right] dy = \int_2^4 \frac{7}{3} y\,dy = \left[\frac{7}{6} y^2\right]_2^4$$

$$= \left[\frac{7}{6} \cdot 16 - \frac{7}{6} \cdot 4\right] = \frac{7}{6} \cdot 12 = \underline{\underline{14}}$$

Dobbeltintegraler over begrensede områder





$$f_A(x,y) = \begin{cases} f(x,y) & \text{når } (x,y)\in A \\ 0 & \text{ellers.} \end{cases}$$

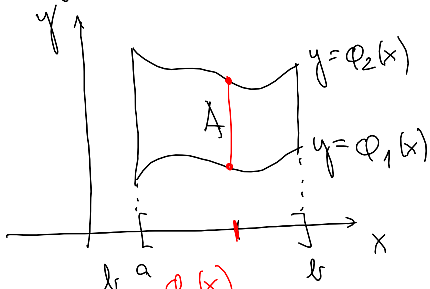
$$\iint_A f(x,y)\,dx\,dy = \underbrace{\overbrace{\iint_R f_A(x,y)\,dx\,dy}^{\text{integrerbar?}}}_{\text{definert allerede}}$$

Definisjon: Anta at A er en begrenset mengde i planet og at $f: A \to \mathbb{R}$ er en funksjon. Da definerer vi

$$\iint_A f(x,y)\,dx\,dy = \iint_R f_A(x,y)\,dx\,dy$$

forutsatt at f_A er integrerbar over R.

Type I:

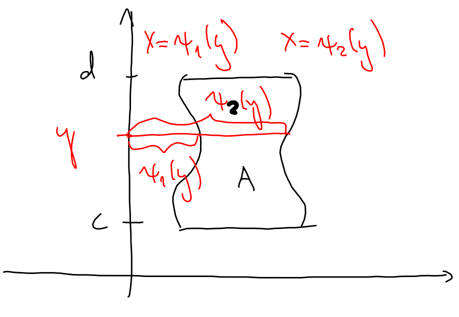


$A = \{(x,y) : a \le x \le b \text{ og } \varphi_1(x) \le y \le \varphi_2(x)\}$

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx$$

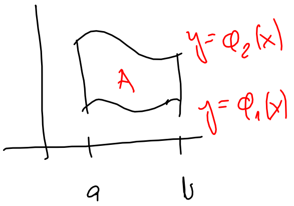
forutsatt at f, φ_1, φ_2 er kontinuerlige, $\varphi_1(x) \le \varphi_2(x)$

Type II



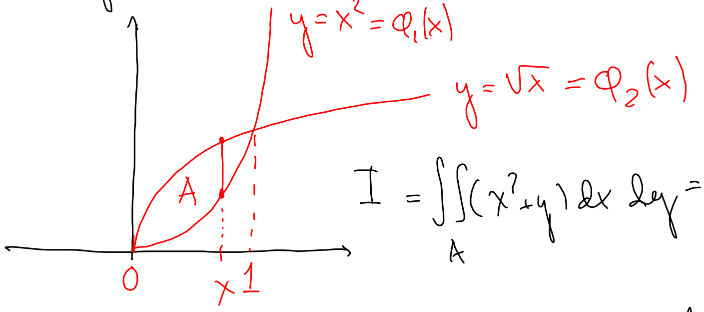
$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy$$

Type I:



$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx$$

Eksempel: Regn ut $\iint_A (x^2+y)\,dx\,dy$ der A er det begrensede området i første kvadrant mellom kurven $y=\sqrt{x}$ og $y=x^2$



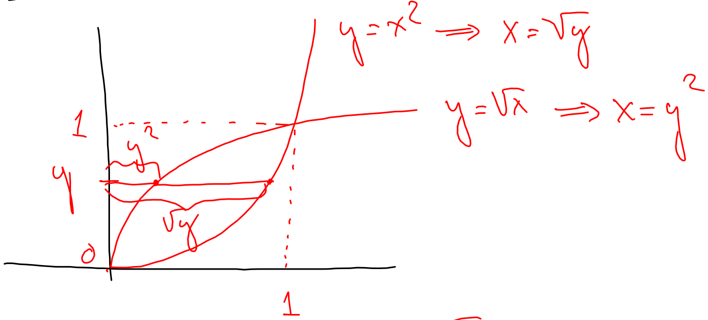
$$I = \iint_A (x^2+y)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (x^2+y)\,dy \right] dx$$

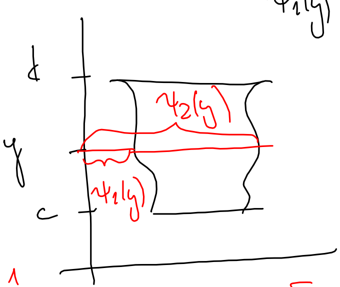
$$= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left[x^2 \underset{}{\overset{x^{1/2}}{\sqrt{x}}} + \frac{x}{2} - x^4 - \frac{x^4}{2} \right] dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^2}{2} - \frac{3}{2}x^4 \right] dx = \dots\dots\dots \quad \text{osv.}$$

Alternativ fremgangstype:

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy$$





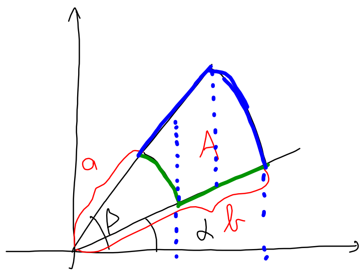
$$I = \iint_A (x^2+y)\,dx\,dy = \int_0^1 \left[\int_{y^2}^{\sqrt{y}} (x^2+y)\,dx \right] dy = \int_0^1 \left(\left[\frac{x^3}{3} + xy \right]_{x=y^2}^{x=\sqrt{y}} \right) dy$$

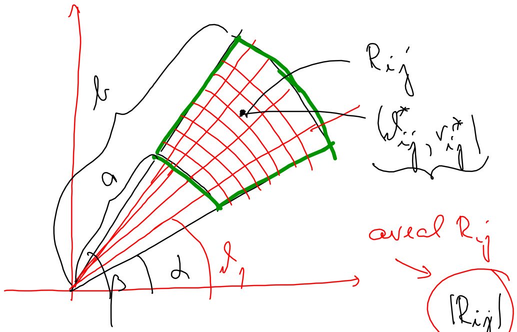
$$= \int_0^1 \left(\left[\frac{y^{3/2}}{3} + y^{3/2} - \frac{y^6}{3} - y^3 \right] \right) dy = \int_0^1 \left(\frac{4}{3} y^{3/2} - \frac{y^6}{3} - y^3 \right) dy$$

$$= \dots\dots$$

Integrasjon i polarkoordinater

$$\iint_A f(x,y)\,dx\,dy$$





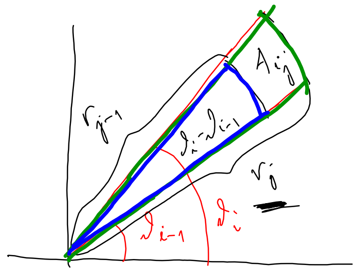
areal R_{ij} → $|R_{ij}|$

Partisjon:

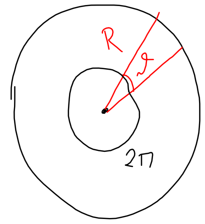
$\alpha=\vartheta_0<\vartheta_1<\vartheta_2<\dots<\vartheta_n=\beta$

$a=r_0<r_1<r_2<\dots<r_m=b$

Volum: $\sum f(r^*_{ij}\cos(\vartheta^*_{ij}), r^*_{ij}\sin(\vartheta^*_{ij}))$



Areal av sirkelsegment:



$$\pi R^2\cdot\frac{\vartheta}{2\pi}= \frac{1}{2}R^2\vartheta$$

$$|A_{ij}| = \frac{1}{2}r_j^2(\vartheta_i-\vartheta_{i-1}) - \frac{1}{2}r_{j-1}^2(\vartheta_i-\vartheta_{i-1})$$

$$= \frac{1}{2}\underbrace{(r_j^2-r_{j-1}^2)}(\vartheta_i-\vartheta_{i-1}) = \frac{(r_j+r_{j-1})}{2}(r_j-r_{j-1})(\vartheta_i-\vartheta_{i-1})$$

$$= r^*_{ij}(r_j-r_{j-1})(\vartheta_i-\vartheta_{i-1})$$

Volum:

$$V \approx \underbrace{\sum f(r^*_{ij}\cos\vartheta^*_{ij}, r^*_{ij}\sin\vartheta^*_{ij})\, r^*_{ij}}_{\text{Riemann}}\,\underbrace{(r_j-r_{j-1})}_{dr}\,\underbrace{(\vartheta_i-\vartheta_{i-1})}_{d\vartheta}$$

Riemann $f(r\cos\vartheta, r\sin\vartheta)\,r$

$$\to \int_\alpha^\beta\int_a^b f(r\cos\vartheta, r\sin\vartheta)\,r\,dr\,d\vartheta$$

Eksempel: $\iint_A xy^2\,dx\,dy$

$$= \int_0^{\pi/2}\int_3^4 (r\cos\vartheta)(r\sin\vartheta)^2\, r\,dr\,d\vartheta$$