Onsdag 28/1: forelesning isteden for plenum.

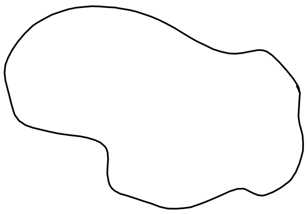
Kjerneregelen

$$f(u_1, \dots, u_m), \quad g_1(x_1, \dots, x_n), g_2(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n)$$

$$h(x_1, \dots, x_n) = f(g_1(x_1, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n))$$

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_i} + \frac{\partial f}{\partial u_2}\frac{\partial g_2}{\partial x_i} + \dots + \frac{\partial f}{\partial u_m}\frac{\partial g_m}{\partial x_i}$$

gassbeholder



Eksempel: $P = f(V, T)$

trykk, volum, temperatur

$V(t)$ volum ved tid t
$T(t)$ temperatur — " —

$P(t) = f(V(t), T(t))$ trykk ved tidspunkt t.

$$\frac{\partial P}{\partial t} = \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial t} + \frac{\partial f}{\partial T}\cdot\frac{\partial T}{\partial t}$$

$$P'(t) = \frac{\partial f}{\partial V} V'(t) + \frac{\partial f}{\partial T} T'(t)$$

$$\boxed{\begin{array}{l} \text{Ideell gass: } PV = kT \\ P = k\frac{T}{V} = f(V,T) \\ \frac{\partial f}{\partial V} = -k\frac{T}{V^2}, \quad \frac{\partial f}{\partial T} = \frac{k}{V} \end{array}}$$

Ideell gass:

$$P'(t) = -k\frac{T}{V^2}V'(t) + \frac{k}{V}T'(t)$$

1.9 Lineæravbildninger

$\vec{T}: \mathbb{R}^n \to \mathbb{R}^m$ kalles en <u>lineæravbildning</u> dersom

(i) $\vec{T}(c\vec{x}) = c\vec{T}(\vec{x})$ for alle $c \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$

(ii) $\vec{T}(\vec{x}+\vec{y}) = \vec{T}(\vec{x}) + \vec{T}(\vec{y})$ for alle $\vec{x}, \vec{y} \in \mathbb{R}^n$

<u>Setning</u>: Hvis $\vec{T}: \mathbb{R}^n \to \mathbb{R}^m$ er en lineæravbildning, så er

$$\vec{T}(c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n) = c_1\vec{T}(\vec{x}_1) + c_2\vec{T}(\vec{x}_2) + \dots + c_n\vec{T}(\vec{x}_n)$$

<u>Bevis</u>: $\vec{T}(\underbrace{c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_{n-1}\vec{x}_{n-1}}_{\vec{x}} + \underbrace{c_n\vec{x}_n}_{\vec{y}}) \overset{(ii)}{=}$

$\vec{T}(c_1\vec{x}_1 + \dots + c_{n-1}\vec{x}_{n-1}) + \vec{T}(c_n\vec{x}_n) \overset{(i)}{=}$

$= \vec{T}(\underbrace{c_1\vec{x}_1 + \dots}_{\vec{x}} + \underbrace{c_{n-1}\vec{x}_{n-1}}_{\vec{y}}) + c_n\vec{T}(\vec{x}_n) \overset{osv.}{=}$

$\vdots \quad \vdots \quad \vdots$

$= c_1\vec{T}(\vec{x}_1) + \dots + c_n\vec{T}(\vec{x}_n)$

<u>Eksempel</u>: La A være en $m \times n$-matrise, og la

$$\vec{T}(\vec{x}) = \underset{\uparrow \mathbb{R}^n}{A\vec{x}} \in \mathbb{R}^m \qquad \vec{T}(\vec{x}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

<u>$\vec{T}$ er lineær:</u>

(i) $\vec{T}(c\vec{x}) = A(c\vec{x}) = cA\vec{x} = c\vec{T}(\vec{x})$

(ii) $\vec{T}(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{T}(\vec{x}) + \vec{T}(\vec{y})$

Hurra, T er lineær!

Setning: Hvis $T:\mathbb{R}^n \to \mathbb{R}^m$ er en lineæravbildning, så finnes det en $m \times n$-matrise A slik at $\vec{T}(\vec{x}) = A\vec{x}$ for alle $\vec{x} \in \mathbb{R}^n$. Den i-te søylen til A er

$$\vec{T}(\vec{e}_i) = T\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-te posisjon.}$$

Bevis: La
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

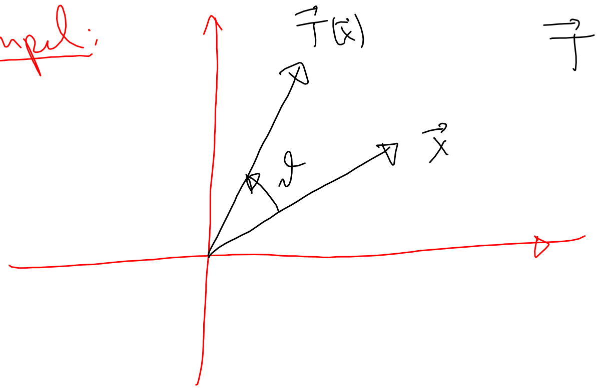
$$= x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

Altså
$$\vec{T}(\vec{x}) = \vec{T}(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1\vec{T}(\vec{e}_1) + x_2\vec{T}(\vec{e}_2) + \cdots + x_n\vec{T}(\vec{e}_n)$$

$$= \underline{x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}} = A\vec{x}.$$

Eksempel:



$\vec T$ drejer enhver vektor en vinkel ϑ.

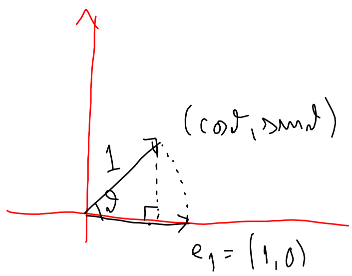
$$\vec T(c\vec x) = c\vec T(\vec x)$$

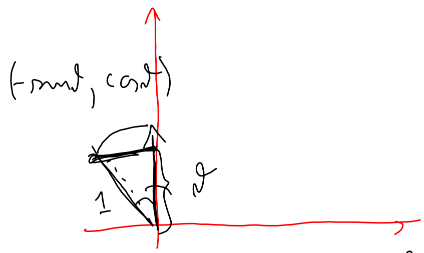
$$\vec T(\vec x+\vec y) = \vec T(\vec x)+\vec T(\vec y)$$

Kan vi finde matricen A_ϑ til T?

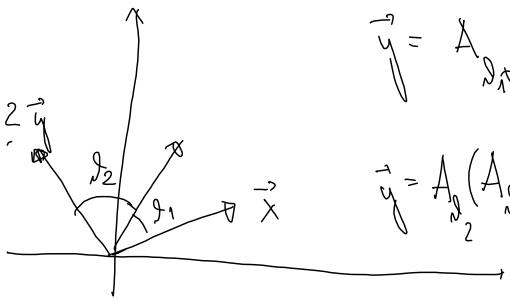
$$A_\vartheta = \left[\vec T(\vec e_1), \vec T(\vec e_2)\right] \qquad \vec e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \vec e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$A_\vartheta = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}$$





$$\vec T(\vec x) = A_\vartheta \vec x = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\cos\vartheta - y\sin\vartheta\\ x\sin\vartheta + y\cos\vartheta\end{pmatrix}$$



$$\vec y = A_{\vartheta_1+\vartheta_2}\vec x = \begin{pmatrix}\cos(\vartheta_1+\vartheta_2) & -\sin(\vartheta_1+\vartheta_2)\\ \sin(\vartheta_1+\vartheta_2) & \cos(\vartheta_1+\vartheta_2)\end{pmatrix}$$

$$\vec y = A_{\vartheta_2}(A_{\vartheta_1}\vec x) = (A_{\vartheta_2}A_{\vartheta_1})\vec x$$

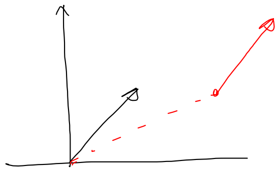
$$\cos(\vartheta_1+\vartheta_2) = \cos\vartheta_2\cos\vartheta_1 - \sin\vartheta_2\sin\vartheta_1$$

$$\sin(\vartheta_1+\vartheta_2) = \cos\vartheta_2\sin\vartheta_1 + \sin\vartheta_2\cos\vartheta_1$$

$$A_{\vartheta_2}A_{\vartheta_1} = \begin{pmatrix}\cos\vartheta_2 & -\sin\vartheta_2\\ \sin\vartheta_2 & \cos\vartheta_2\end{pmatrix}\begin{pmatrix}\cos\vartheta_1 & -\sin\vartheta_1\\ \sin\vartheta_1 & \cos\vartheta_1\end{pmatrix}$$

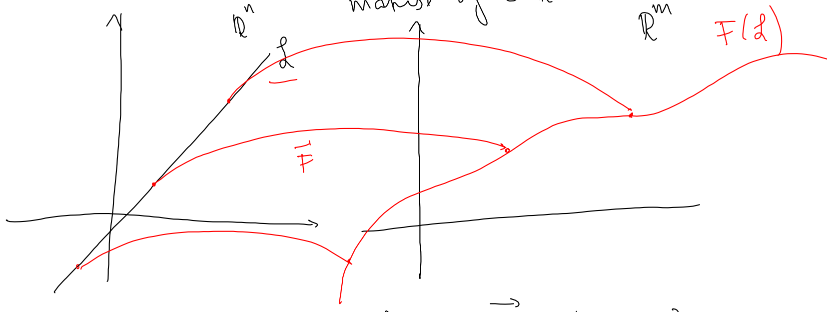
$$= \begin{pmatrix}\cos\vartheta_2\cos\vartheta_1 - \sin\vartheta_2\sin\vartheta_1 & -\cos\vartheta_2\sin\vartheta_1 - \sin\vartheta_2\cos\vartheta_1\\ & \end{pmatrix}$$

1.10 Affinavbildninger



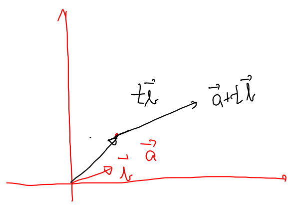
$\vec{T}(\vec{0}) = \vec{T}(0\vec{0}) = 0\vec{T}(\vec{0}) = \vec{0}$

Definisjon: En affinavbildning $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ er en funksjon på formen $\vec{F}(\vec{x}) = A\vec{x} + \vec{c}$ der A er en $m \times n$-matrise og $\vec{c} \in \mathbb{R}^m$



Rett linje gjennom $\vec{a}$ i retningen $\vec{b}$: $\vec{r}(t) = \vec{a} + t\vec{b}$

$\vec{F}(\vec{r}(t))$



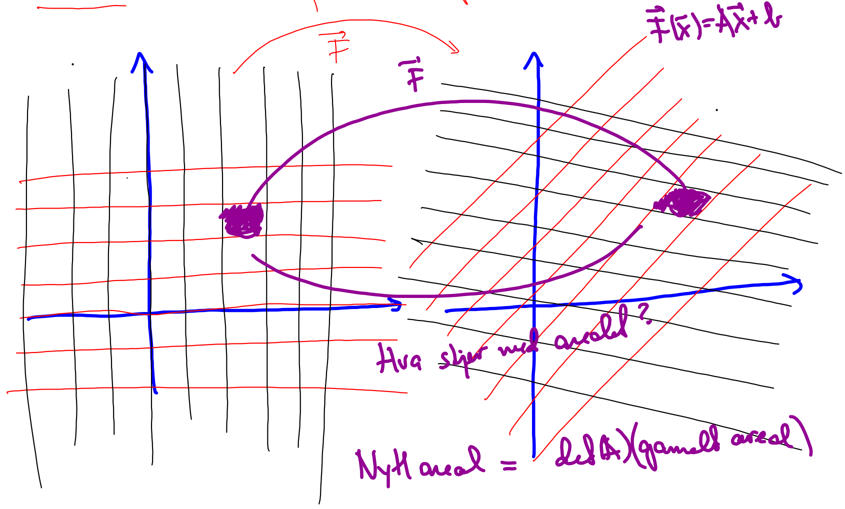
Setning: Anta at $\mathcal{L}$ er linjen $\vec{r}(t) = \vec{a} + t\vec{b}$ i $\mathbb{R}^n$, og at $\vec{F}(\vec{x}) = A\vec{x} + \vec{c}$ er en affinavbildning fra $\mathbb{R}^n$ til $\mathbb{R}^m$. Hvis $A\vec{b} \neq \vec{0}$, så er bildet av $\mathcal{L}$ en rett linje med retningsvektor $A\vec{b}$.

Bevis: $\vec{F}(\vec{r}(t)) = A\vec{r}(t) + \vec{c} = A(\vec{a} + t\vec{b}) + \vec{c} = A\vec{a} + tA\vec{b} + \vec{c}$

$$= \underbrace{(A\vec{a} + \vec{c})}_{} + t\underbrace{A\vec{b}}_{}$$

linjen gjennom $A\vec{a} + \vec{c}$ i retning $A\vec{b}$

Observer: Parallelle linjer avbildes på parallelle linjer.



Hva skjer med arealet?

Nytt areal $= \det(A)$ (gamle areal)