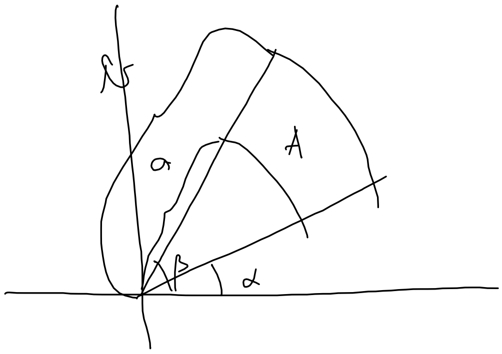
Dobbeltintegraler i polarkoordinater

$$\iint_A f(x,y)\,dx\,dy$$

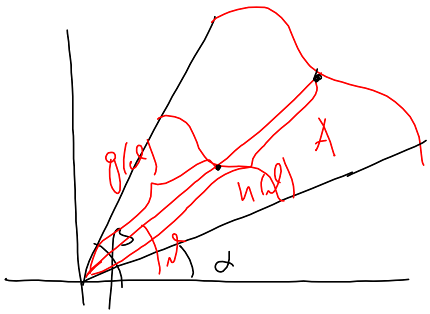
$$= \int_a^b \left[\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\right] dr$$

$$= \int_\alpha^\beta \left[\int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta$$



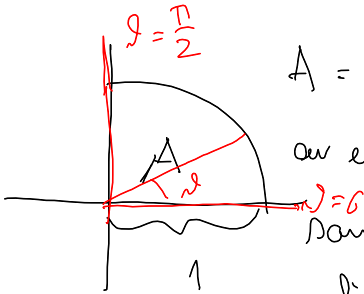
Generalisering:

$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \left[\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, \underline{r}\, dr\right] d\theta$$



Eksempel:

$$\iint_A \overset{r\cos\theta}{x}\, dx\,dy$$



A = den delen av enhetssirkelen som ligger i første kvadrant.

$$= \int_0^1 \left[\int_0^{\pi/2} r\cos\theta \; r\, d\theta\right] dr$$

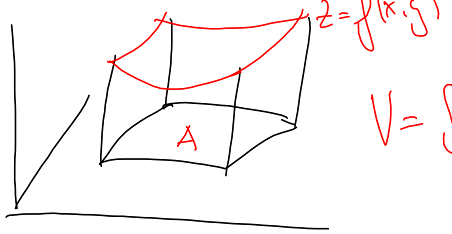
$$= \int_0^1 \left[\int_0^{\frac{\pi}{2}} r^2 \cos\theta\, d\theta\right] dr = \int_0^1 r^2 \left[\sin\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} dr$$

$$= \int_0^1 r^2 \left[1 - 0\right] dr = \int_0^1 r^2\, dr = \left[\frac{r^3}{3}\right]_0^1 = \underline{\underline{\frac{1}{3}}}$$

Anvendelser av dobbeltintegraler

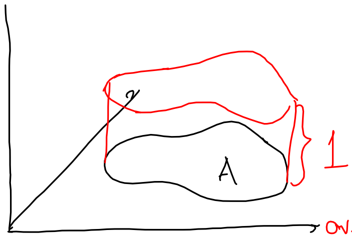
Hva kan vi bruke dobbeltintegraler til å regne ut?

Volumer: $f \geq 0$



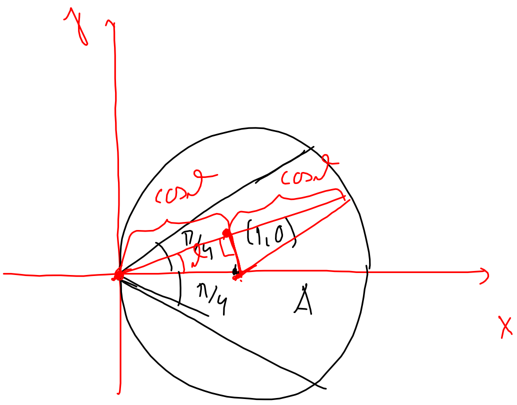
$$V = \iint_A f(x,y)\,dx\,dy$$

Arealer:



$$V = \iint_A 1\,dx\,dy = \text{areal} \cdot 1$$

$$\text{areal}(A) = \iint_A 1\,dx\,dy.$$



$$\text{areal} = \iint_A 1\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{2\cos\theta} 1 \cdot \underbrace{r}_{r}\,dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4\cos^2\theta}{2}\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2\theta\,d\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

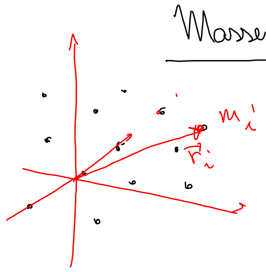
$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\cos^2\theta = \cos 2\theta + 1$$

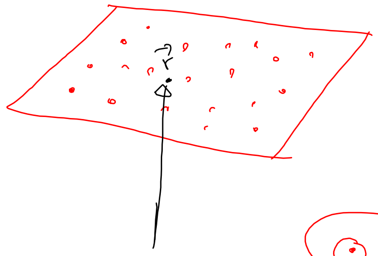
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta + 1)\,d\theta = \left[\frac{1}{2}\sin 2\theta + \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$

$$= \left[\left(\frac{1}{2}\underbrace{\sin\frac{\pi}{2}}_{1} + \frac{\pi}{4} \right) - \left(\frac{1}{2}\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1} - \frac{\pi}{4} \right) \right] = \underline{\underline{1 + \frac{\pi}{2}}}$$

Massemiddelpunkt (massesenter)



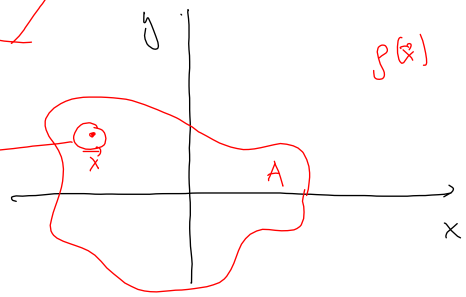
$$\vec r = \frac{\sum_{i=1}^{n} m_i \vec r_i}{\sum_{i=1}^{n} m_i} \qquad \text{massemiddelpunkt (senter)}$$

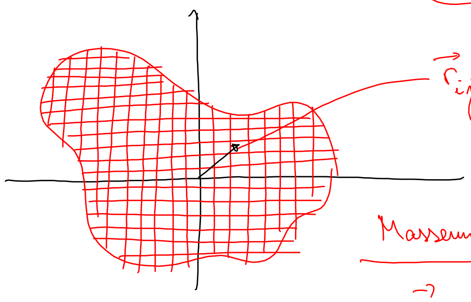


Hva er massemiddelpunktet til en hanfinnelige flate?

$\rho(\vec x)$

massen ≈ arealet · $\rho(x)$





$\vec r_{ij}$ $\quad A_{ij}$ areal $\quad \rho(\vec r_{ij})$

$$m_{ij} = \rho(\vec r_{ij}) A_{ij}$$

Massemiddelpunkt:

$$\vec r \approx \frac{\sum m_{ij} \vec r_{ij}}{\sum m_{ij}} = \frac{\sum \rho(\vec r_{ij}) \vec r_{ij} A_{ij}}{\sum \rho(\vec r_{ij}) A_{ij}}$$

(x_{ij}, y_{ij})

$$\sum \rho(\vec r_{ij}) A_{ij} = \underbrace{\sum_{i,j} \rho(x_{ij}, y_{ij})(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{Riemannsum for } \rho} \longrightarrow \underbrace{\iint_A \rho(x,y)\,dx\,dy}_{\text{massen til flaten.}}$$

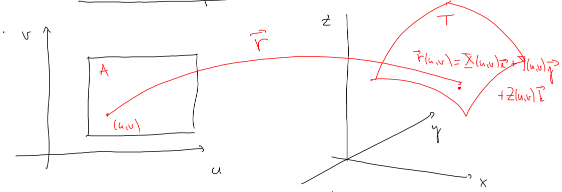
$$\sum \rho(x_{ij}, y_{ij})\, x_{ij} A_{ij} = \underbrace{\sum_{i,j} \rho(x_{ij}, y_{ij})\, x_{ij} (x_i - x_{i-1})(y_j - y_{j-1})}_{\text{Riemannsum for } x\rho} \rightarrow \iint x \rho(x,y)\,dx\,dy$$

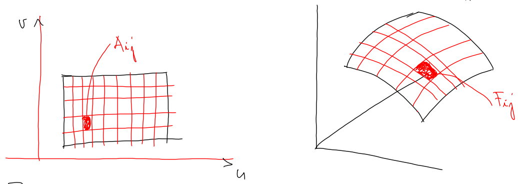
$$\sum \rho(x_{ij}, y_{ij})\, y_{ij} A_{ij} \xrightarrow{\hspace{3cm}} \iint y \rho(x,y)\,dx\,dy$$

Massemiddelpunktet er definert som punktet $\bar{r} = (\bar x, \bar y)$ der

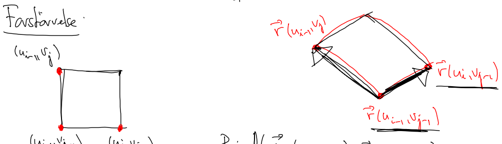
$$\bar x = \frac{\iint_A x \rho(x,y)\,dx\,dy}{\iint_A \rho(x,y)\,dx\,dy} \qquad \bar y = \frac{\iint_A y \rho(x,y)\,dx\,dy}{\iint_A \rho(x,y)\,dx\,dy}$$

Arealet av flater





Forståelse:



$$P_{ij} = \Big|\big(\underbrace{\vec r(u_i,v_{j-1}) - \vec r(u_{i-1},v_{j-1})}_{\frac{\partial \vec r}{\partial u}(u_i-u_{i-1})}\big) \times \big(\underbrace{\vec r(u_{i-1},v_j) - \vec r(u_{i-1},v_{j-1})}_{\frac{\partial \vec r}{\partial v}(v_j-v_{j-1})}\big)\Big|$$

$$\approx \left|\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right|(u_i-u_{i-1})(v_j-v_{j-1})$$

Flatearealet $\approx \sum P_{ij} = \underbrace{\sum_{i,j}\left|\left(\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right)(u_{i-1},v_{j-1})\right|(u_i-u_{i-1})(v_j-v_{j-1})}_{\text{Riemannsum for } \left|\left(\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right)(u,v)\right|}$

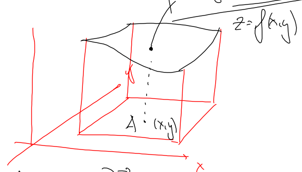
$$\to \iint_A \left|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}(u,v)\right| du\,dv$$

Definisjon: Arealet til flaten parametrisert av $\vec r: A \to \mathbb{R}^3$ er

$$\iint_A \left|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}(u,v)\right| du\,dv$$

$\boxed{\vec r(x,y) = x\vec i + y\vec j + f(x,y)\vec k}$

Spesialtilfelle:



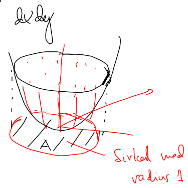
$$\frac{\partial \vec r}{\partial x} = \vec i + \frac{\partial f}{\partial x}\vec k \qquad \frac{\partial \vec r}{\partial y} = \vec j + \frac{\partial f}{\partial y}\vec k$$

$$\frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y} = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y}\end{vmatrix} = -\frac{\partial f}{\partial x}\vec i - \frac{\partial f}{\partial y}\vec j + \vec k$$

$$\left|\frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y}\right| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

Areal: $\displaystyle\iint_A \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\,dy$

Eksempel: $z = x^2+y^2$



Arealet $= \displaystyle\iint_A \sqrt{(2x)^2+(2y)^2+1}\,dx\,dy$

$$= \iint_A \sqrt{\underbrace{4x^2+4y^2}_{\substack{4(x^2+y^2)\\4r^2}}+1}\,dx\,dy = \int_0^1\left[\int_0^{2\pi}\sqrt{4r^2+1}\,r\,d\theta\right]dr$$

$u = 4r^2+1$

$du = 8r\,dr$

$r\,dr = \frac{1}{8}du$

$$= \int_0^1 \sqrt{4r^2+1}\,r\left[\underbrace{\int_0^{2\pi} 1\,d\theta}_{2\pi}\right]dr = 2\pi\int_0^1\sqrt{4r^2+1}\,r\,dr$$

$r=0: u=1$

$r=1: u=5$

$$= 2\pi\int_1^5 \sqrt{u}\,\frac{1}{8}du = \frac{\pi}{4}\int_1^5 u^{1/2}\,du$$

$$= \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=5} =$$