Greens teorem

Linjeintegral: $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

I planet: $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j} = P(x,y)\vec{i} + Q(x,y)\vec{j}$

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

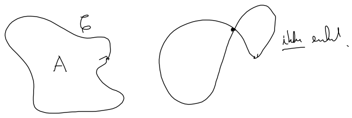
Dermed:

$$\int_{\mathcal{C}} \vec{F}\, d\vec{r} = \int_a^b \left(P(x(t),y(t))\vec{i} + Q(x(t),y(t))\vec{j} \right) \cdot (x'(t)\vec{i} + y'(t)\vec{j})\, dt$$

$$= \int_a^b [P(x(t),y(t))\underbrace{x'(t)}_{dx} + Q(x(t),y(t))\underbrace{y'(t)}_{dy}]\, dt$$

$$= \int_{\mathcal{C}} P\,dx + Q\,dy$$

$\mathcal{C}$: enkel, lukket kurve: Starter og ender i samme punkt, uten skjæringer underveis:

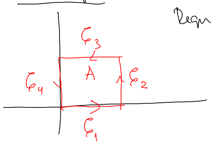


Greens teorem: Anta at $\mathcal{C}$ er en enkel, lukket kurve i planet som er stykkevis glatt og omkranser et område A. Da er

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

for alle P og Q som har kontinuerlig partiellderiverte i et område som inneholder A. $\mathcal{C}$ må være orientert mot klokka.

Eksempel: La $\mathcal{C}$ være kurven som omslutter enhetskvadratet:



Regn ut

$$\int_{\mathcal{C}} P\,dx + Q\,dy$$

når $P = (x+y)$, $Q = x^2 y$, dvs

$$\int_{\mathcal{C}} (x+y)\,dx + x^2 y\,dy$$

Ifølge Greens teorem:

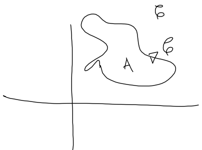
$$\int_{\mathcal{C}} \underbrace{(x+y)}_{P}\, dx + \underbrace{x^2y}_{Q}\, dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_A (2xy - 1)\,dx\,dy$$

$$= \iint_A 2xy\,dx\,dy - \overbrace{\iint_A 1\,dx\,dy}^{1} = \int_0^1 \left[\int_0^1 2xy\,dy \right] dx - 1$$

$$= \int_0^1 \left[xy^2 \right]_0^{y=1} dx - 1 = \int_0^1 x\,dx - 1 = \left[\frac{x^2}{2} \right]_0^1 - 1 = \underline{\underline{-\frac{1}{2}}}$$

Korollar: Anta at $\mathcal{C}$ er en stykkevis glatt, enkel, lukket kurve som omkranser et område A. Da er arealet til A lik

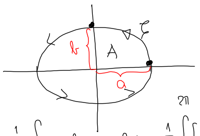
$$\text{areal}(A) = \int_{\mathcal{C}} x\,dy = -\int_{\mathcal{C}} y\,dx = \frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy$$



Bevis: $\frac{1}{2}\int_{\mathcal{C}} \underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy = \frac{1}{2}\iint_A \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dx\,dy$

$= \frac{1}{2}\iint_A (1+1)\,dx\,dy = \iint_A 1\,dx\,dy = \text{areal}(A).$

Eksempel:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\vec{r}(t) = \underbrace{a\cos t}_{x(t)}\,\vec{i} + \underbrace{b\sin t}_{y(t)}\,\vec{j}$

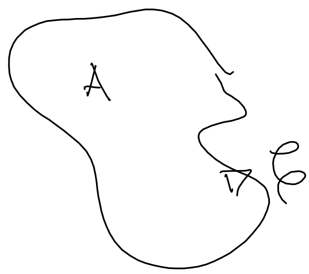
$t \in [0, 2\pi]$

$$\text{areal}(A) = \frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} [-b\sin t\,(-a\sin t)\,dt + a\cos t\, b\cos t\,dt]$$

$$= \frac{ab}{2}\int_0^{2\pi} [\sin^2 t + \cos^2 t]\,dt = \frac{ab}{2}\underbrace{\int_0^{2\pi} 1\,dt}_{2\pi} = \frac{ab}{2}\cdot 2\pi = \underline{\underline{\pi a b}}$$

Litt om bevis

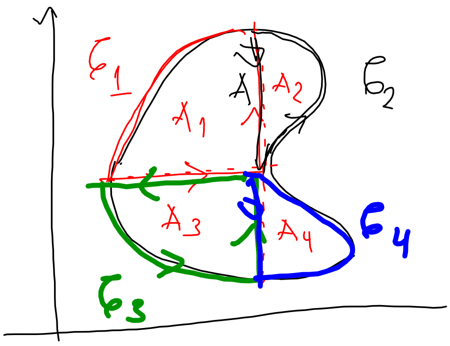
$$\int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$



To biter:

$$\int_C P\,dx = -\iint_A \frac{\partial P}{\partial y}\,dx\,dy$$

$$\int_C Q\,dy = \iint_A \frac{\partial Q}{\partial x}\,dx\,dy$$



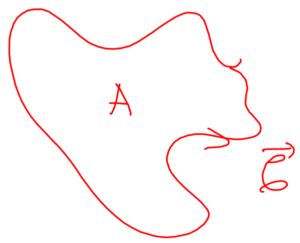
Fra enkle til kompliserte områder:

Anta at Greens teorem holder for områdene A_1, A_2, A_3, A_4. Da kan vi også vise det for A.

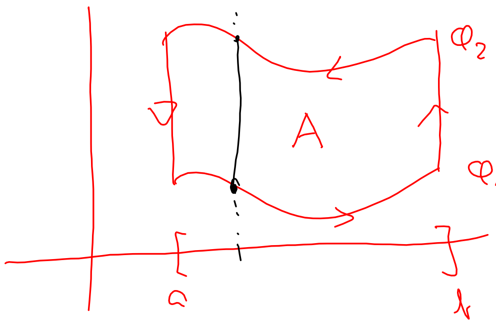
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \iint_{A_1} + \iint_{A_2} + \iint_{A_3} + \iint_{A_4}$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



Plan: Beviser P-delen for områder av type I

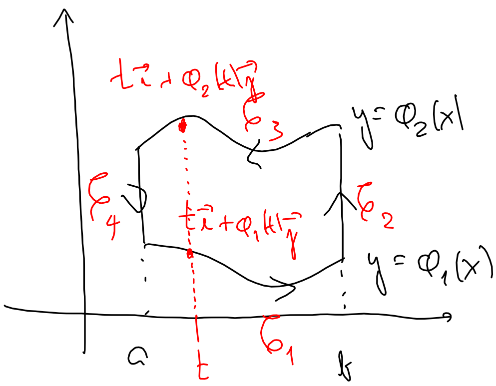


$$\int_{\mathcal{C}} P\,dx = \iint_A -\frac{\partial P}{\partial y}\,dx\,dy$$

Regner først ut:
$$\iint_A \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\right] dx$$

$$= \int_a^b \left[P(x,y)\right]_{y=\varphi_1(x)}^{y=\varphi_2(x)} dx = \int_a^b \left[P(x,\varphi_2(x)) - P(x,\varphi_1(x))\right] dx$$

Regner så ut:
$$\int_{\mathcal{C}} P\,dx = \int_{\mathcal{C}_1} P\,dx + \underbrace{\int_{\mathcal{C}_2} P\,dx}_{0} + \int_{\mathcal{C}_3} P\,dx + \underbrace{\int_{\mathcal{C}_4} P\,dx}_{0}$$



$\vec{r}_1(t) = t\vec{i} + \varphi_1(t)\vec{j}$, $t \in [a,b]$

$$\int_{\mathcal{C}_1} P\,dx = \int_a^b P(t,\varphi_1(t))\,dt$$

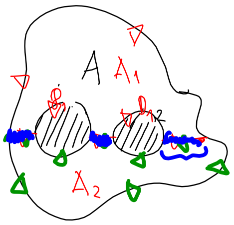
OBS: Wrong way.

$\vec{r}_3(t) = t\vec{i} + \varphi_2(t)\vec{j}$, $t \in [a,b]$

$$\int_{\mathcal{C}_3} P\,dx = -\int_a^b P(t,\varphi_2(t))\,dt$$

I alt:
$$\int_{\mathcal{C}} P\,dx = \int_a^b P(t,\varphi_1(t))\,dt - \int_a^b P(t,\varphi_2(t))\,dt = -\iint_A \frac{\partial P}{\partial y}\,dx\,dy$$

Greens teorem for områder med hull



$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \text{linjeintegral ?}$$

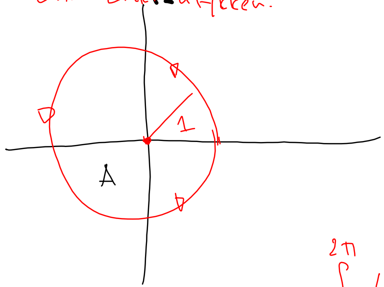
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{A_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \iint_{A_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_{C} P\,dx + Q\,dy - \int_{D_1} P\,dx + Q\,dy - \int_{D_2} P\,dx + Q\,dy$$

Eksempel: $\vec{F}(x,y) = \underbrace{-\frac{y}{x^2+y^2}}_{P(x,y)}\vec{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q(x,y)}\vec{j} \qquad (x,y) \neq (0,0)$

Etter trikkeulykken:



$$\vec{r}(t) = \overbrace{\cos t}^{x(t)}\vec{i} + \overbrace{\sin t}^{y(t)}\vec{j}, \quad t \in [0, 2\pi]$$

$dx = -\sin t\,dt \qquad dy = \cos t\,dt$

$$\int_C \vec{F}\,d\vec{r} = \int_C P\,dx + Q\,dy$$

$$= \int_0^{2\pi} \left(-\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t)\,dt + \frac{\cos t}{\cos^2 t + \sin^2 t}\cos t\,dt \right)$$

$$= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

Prøver med Greens (overser hullet i midten):

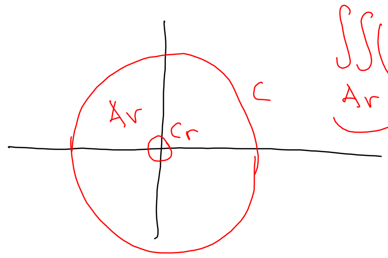
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - \overbrace{x \cdot 2x}^{-2x^2}}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$= \iint_A 0\,dx\,dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = -\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Green virker ikke pga. singulariteten i origo



$$\underbrace{\iint_{A_r}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{0} = \underbrace{\int_C P\,dx + Q\,dy}_{2\pi} - \underbrace{\int_{C_r} P\,dx + Q\,dy}_{2\pi}$$