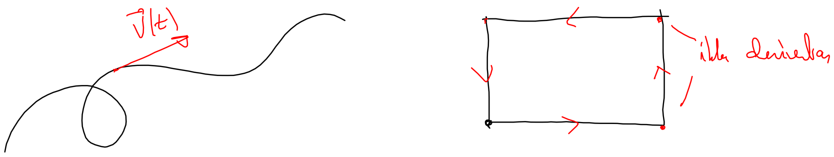
Kurver

$\vec{r}: [a,b] \to \mathbb{R}^n$ kontinuerlig, $\vec{r}\,'(t)$

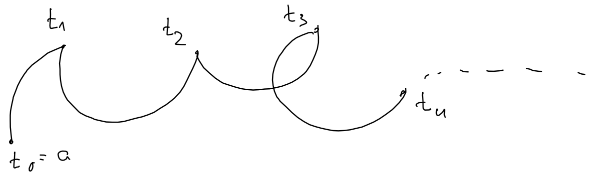
Definisjon: En parametrisert kurve $\vec{r}: [a,b] \to \mathbb{R}^n$ kalles glatt dersom den deriverte $\vec{r}\,'(t)$ eksisterer for alle $t \in (a,b)$ og er kontinuerlig på dette intervallet.



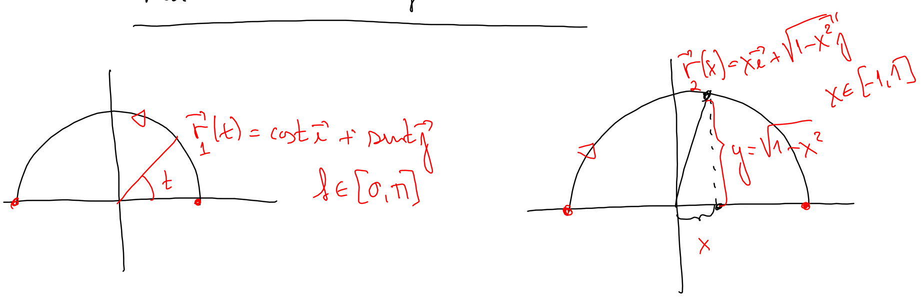
Definisjon: En parametrisert kurve $\vec{r}: [a,b] \to \mathbb{R}^n$ kalles stykkevis glatt dersom den er kontinuerlig og det finnes en oppdeling

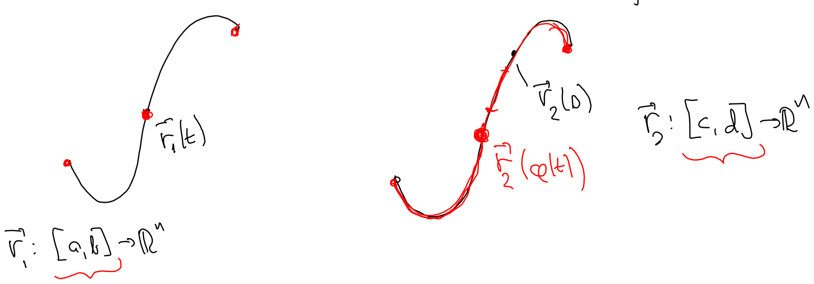
$$a = t_0 < t_1 < t_2 < \dots < t_m = b$$

av $[a,b]$ slik at $\vec{r}: [t_{i-1}, t_i] \to \mathbb{R}$ er en glatt kurve.



Når er to kurver geometrisk like?





Definisjon: To parametriserte kurver $\vec{r}_1: [a,b] \to \mathbb{R}^n$ og $\vec{r}_2: [c,d] \to \mathbb{R}^n$ kalles ekvivalente dersom det finnes en funksjon

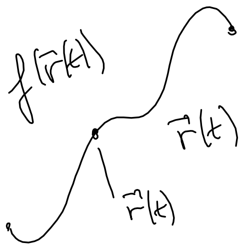
$\varphi: [a,b] \to [c,d]$ slik at

(i) $\vec{r}_1(t) = \vec{r}_2(\varphi(t))$ for alle $t \in [a,b]$.

(ii) φ er strengt monoton og $\varphi([a,b]) = [c,d]$

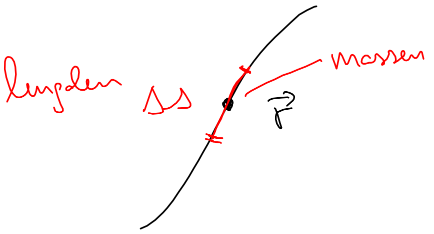
(iii) φ er deriverbar i alle punkter $t \in (a,b)$ og $\varphi'(t) \neq 0$.

Linjeintegraler for skalarfunksjoner



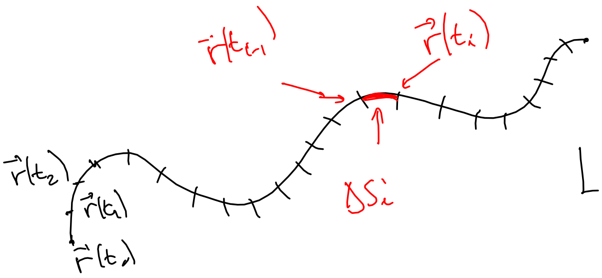
$$\int_C f\,ds$$

Tenk at kurven er en tråd med varierende tetthet og at vi skal regne ut den totale massen.



$$\Delta m = f(\vec r)\Delta s$$

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$



$$m_i = f(\vec r(t_{i-1}))\underbrace{\Delta s_i}_{?}$$

$$L(a,t) = \int_a^t \underbrace{\sqrt{x_1'(s)^2 + \dots + x_n'(s)^2}}_{v(s)}\,ds = \int_a^t v(s)\,ds$$

$$\Delta(s_i) = L(a,t_i) - L(a,t_{i-1}) = \int_{t_{i-1}}^{t_i} v(s)\,ds \approx v(t_{i-1})(t_i - t_{i-1})$$

$$m_i \approx f(\vec r(t_{i-1}))\,v(t_{i-1})(t_i - t_{i-1})$$

Total masse: $M = \sum m_i \approx \underbrace{\sum f(\vec r(t_{i-1}))\,v(t_{i-1})(t_i - t_{i-1})}_{\text{Riemannsum } f(\vec r(t))v(t)}$

$$\to \int_a^b f(\vec r(t))\,v(t)\,dt$$

Definisjon: Anta at $\vec{r}:[a,b]\to\mathbb{R}^n$ er en stykkevis glatt kurve $\mathcal{C}$ og at $f:\mathbb{R}^n\to\mathbb{R}$ er en kontinuerlig funksjon. Da definerer vi linjeintegralet $\int_{\mathcal{C}} f\,ds$ ved

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(\vec{r}(t))\,v(t)\,dt$$

Hvorfor $\mathcal{C}$?: Fordi dersom $\vec{r}_1$ og $\vec{r}_2$ er to forskjellige men ekvivalente parametriseringer, så

$$\int_a^b f(\vec{r}_1(t))\,v_1(t)\,dt = \int_c^d f(\vec{r}_2(t))\,v_2(t)\,dt$$

Eksempel: La $\vec{r}(t)=\underbrace{t}_{x(t)}\vec{i}+\underbrace{t^2}_{y(t)}\vec{j}$, $t\in[0,1]$, $\underline{f(x,y)=xy}$

$$\vec{v}(t)=\vec{r}\,'(t)=\vec{i}+2t\vec{j},\quad v(t)=|\vec{v}(t)|=\sqrt{1^2+(2t)^2}=\sqrt{1+4t^2}$$

$$\int_{\mathcal{C}} f\,ds = \int_0^1 f(\vec{r}(t))\,v(t)\,dt = \int_0^1 t\cdot t^2\sqrt{1+4t^2}\,dt = \int_0^1 t^3\sqrt{1+4t^2}\,dt$$

$$\boxed{u=1+4t^2}$$

$$du=8t\,dt$$

$$\frac{1}{8}du=t\,dt$$

$$t^2=\frac{u-1}{4}$$

$$=\int_1^5 t^2\sqrt{u}\,\frac{1}{8}du = \int_1^5 \frac{u-1}{4}\sqrt{u}\,\frac{1}{8}du$$

$$=\frac{1}{32}\int_1^5\left(u^{3/2}-u^{1/2}\right)du = \ldots\ldots$$

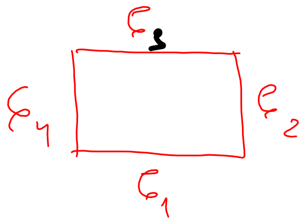
Regneregler: (i) $\int_C af\,ds = a\int f\,ds$

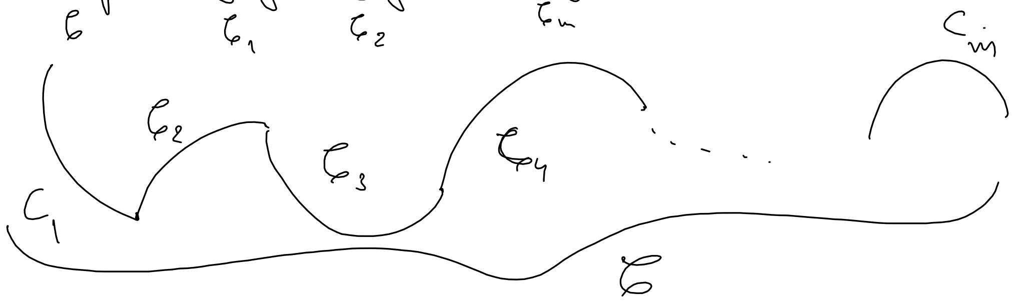
(ii) $\int_C (f+g)\,ds = \int_C f\,ds + \int_C g\,ds$

(iii) $\int_C (f-g)\,ds = \int_C f\,ds - \int_C g\,ds$

(iv) $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_n} f\,ds.$

$\int_C f\,ds = \int_{C_1} f\,ds + \dots$





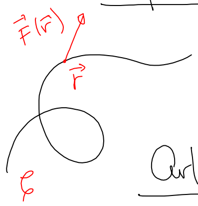
Teorem: Dersom $\vec{r}_1$ og $\vec{r}_2$ er to ekvivalente parametriseringer, så er

$$\int_{C_1} f\,ds = \int_{C_2} f\,ds$$

"Bare kurven teller, ikke parametriseringen"

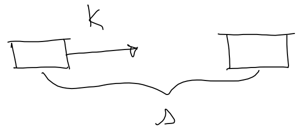
parametrisert med $\vec{r}_1$ parametrisert med $\vec{r}_2$

Linjeintegraler for vektorfelt.

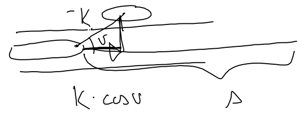


$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$$

Arbeid = kraft × vei

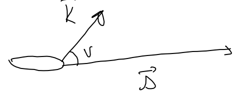


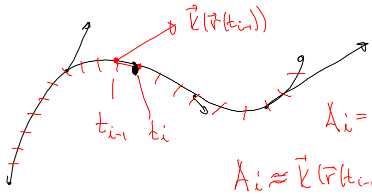
$A = K\cdot s$



$A = K\cos v\cdot s$

$$A = |\vec{K}||\vec{s}|\cos v = \vec{K}\cdot\vec{s}$$





$$A_i = \vec{K}(\vec{r}(t_{i-1}))\cdot(\vec{r}(t_i)-\vec{r}(t_{i-1}))$$

$$A_i \approx \vec{K}(\vec{r}(t_{i-1}))\cdot\vec{r}\,'(t_{i-1})\,(t_i-t_{i-1})$$

Totalt arbeid: $A = \sum A_i = \sum \underbrace{\vec{K}(\vec{r}(t_{i-1}))\cdot\vec{r}\,'(t_{i-1})}\,(t_i-t_{i-1})$

Riemannsum

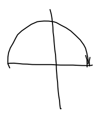
$\vec{K}(\vec{r}(t))\cdot\vec{r}\,'(t)$

$$\to \int_a^b \vec{K}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

Definisjon: Anta at $\vec{r}:[a,b]\to\mathbb{R}^n$ er en stykkevis glatt kurve og at $\vec{F}:\mathbb{R}^n\to\mathbb{R}^n$ er kontinuerlig. Da er linjeintegralet $\int_{\mathcal{C}}\vec{F}\cdot d\vec{r}$ definert ved

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt$$

Eksempel: $\vec{r}(t) = \overbrace{\cos t}^{x(t)}\,\vec{i} + \overbrace{\sin t}^{y(t)}\,\vec{j}$, $t\in[0,\pi]$



$\vec{F}(x,y) = -y\,\vec{i} + xy\,\vec{j}$

$\vec{v}(t) = \vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} = \int_a^b(-\sin t\,\vec{i} + \cos t\sin t\,\vec{j})\cdot(-\sin t\,\vec{i} + \cos t\,\vec{j})$$

$$= \int_a^b[\sin^2 t + \underbrace{\cos^2 t\sin t}_{u=\cos t}]\,dt = \ldots\ldots\ldots$$