UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2440 — Ordinary Differential Equations and Optimal Control Theory

Solution

Problem 1

The equilibrium points are solution of the system

$$-3x + 4y + xy = 0 \quad (1)$$
$$-2x + 6y - xy = 0. \quad (2)$$

Adding up the two equations, we get

$$-5x + 10y = 0$$

which yields $x = 2y$. Hence, after plugging this in (1), we get

$$-6y + 4y + 2y^2 = 0$$

which gives $y(2y - 2) = 0$. There are two equilibrium points: $(0, 0)$ and $(2, 1)$. Let $f_1(x, y) = -3x + 4y + xy$ and $f_2(x, y) = -2x + 6y - xy$. The linearization of the system around the equilibrium point $Y_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is given by $Z' = JZ$ where

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

and $Z = Y - Y_0$. Here, we obtain

$$J = \begin{pmatrix} -3 + y & 4 + x \\ -2 - y & 6 - x \end{pmatrix}.$$

For $(0, 0)$, it yields

$$J = \begin{pmatrix} -3 & 4 \\ -2 & 6 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\lambda^2 - 3\lambda - 10 = 0$$

There are two distinct eigenvalues

$$\lambda = \frac{3 \pm 7}{2},$$

which have opposite signs. The equilibrium is a saddle point. For the equilibrium $(2, 1)$, we obtain

$$J = \begin{pmatrix} -2 & 6 \\ -3 & 4 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\lambda^2 - 2\lambda + 10 = 0$$

and we obtain two complex eigenvalues

$$\lambda = \frac{2 \pm 6i}{2} = 1 \pm 3i.$$

Since $\mathrm{Re}(\lambda) > 0$, it corresponds to a spiral source.

Problem 2 (weight 40%)

2a (weight 20%)

The matrix A has two complex eigenvalues $\lambda = 1 \pm 3i$ (see problem 1). A complex eigenvector associated with $\lambda = 1 + 3i$ satisfies

$$\begin{pmatrix} -3 - 3i & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

so that $u = \begin{pmatrix} 2 \\ 1 + i \end{pmatrix}$ is an eigenvector. Two independent solutions are given by $\mathrm{Re}(e^{\lambda t} u)$ and $\mathrm{Im}(e^{\lambda t} u)$. We have

$$\begin{aligned} e^{\lambda t} u &= e^t (\cos(3t) + i \sin(3t)) \begin{pmatrix} 2 \\ 1 + i \end{pmatrix} \\ &= e^t \begin{pmatrix} 2\cos(3t) \\ \cos(3t) - \sin(3t) \end{pmatrix} + ie^t \begin{pmatrix} 2\sin(3t) \\ \cos(3t) + \sin(3t). \end{pmatrix} \end{aligned}$$

Hence, the general solution is

$$Y(t) = Ae^t \begin{pmatrix} 2\cos(3t) \\ \cos(3t) - \sin(3t) \end{pmatrix} + Be^t \begin{pmatrix} 2\sin(3t) \\ \cos(3t) + \sin(3t). \end{pmatrix} \tag{3}$$

for any constant A and B.

2b (weight 10%)

We have to determine A and B in (3) such that

$$Y(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = A \begin{pmatrix} 2 \\ 1 \end{pmatrix} + B \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We get $A = B = 1$ and

$$Y(t) = e^t \begin{pmatrix} 2\cos(3t) + 2\sin(3t) \\ 2\cos(3t) \end{pmatrix}.$$

2c (weight 10%)

A fundamental matrix solution is given by

$$\Phi(t) = e^t \begin{pmatrix} 2\cos(3t) & 2\sin(3t) \\ \cos(3t) - \sin(3t) & \cos(3t) + \sin(3t) \end{pmatrix}$$

The exponential matrix e^{tA} is equal to

$$e^{tA} = \Phi(t)\Phi(0)^{-1}.$$

We compute $\Phi(0)$ and $\Phi(0)^{-1}$. We obtain

$$\Phi(0) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

Hence,

$$e^{tA} = e^t \begin{pmatrix} \cos(3t) - \sin(3t) & 2\sin(3t) \\ -\sin(3t) & \cos(3t) + \sin(3t) \end{pmatrix}$$

Problem 3 (weight 40%)

3a (weight 10%)

The matrix $\begin{pmatrix} 1 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$ has two distincs eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$. The vectors

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}.$$

are eigenvectors for λ_1 and λ_2, respectively. The general solution is

$$Y(t) = Ae^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{-2t} \begin{pmatrix} 1 \\ -6 \end{pmatrix}. \tag{4}$$

3b (weight 20%)

The Hamiltonian is given by

$$H(t, x, u, p) = -3x^2 - u^2 + p(x + u).$$

The function $u \mapsto H$ is concave so that $\frac{\partial H}{\partial u} = 0$ is a necessary and sufficient condition for a maximizer. We have

$$\frac{\partial H}{\partial u} = -2u + p$$

and, by the maximum principle, $u^* = \frac{p}{2}$. We have

$$\dot{x} = x + u = x + \frac{p}{2}$$

and, from the maximum principle,

$$\dot{p} = -\frac{\partial H}{\partial x} = 6x - p.$$

Hence $Y(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$ satisfies the ordinary differential equation given in **3a** and it follows from there that

$$x(t) = Ae^{2t} + Be^{-2t},$$
$$p(t) = 2Ae^{2t} - 6Be^{-2t}.$$

Since $x(0) = 0$, we get $A = -B$ and the system above rewrites

$$x(t) = A(e^{2t} - e^{-2t}), \tag{5a}$$
$$p(t) = A(2e^{2t} + 6e^{-2t}). \tag{5b}$$

We have to determine A. Since $x(\ln(2)) = -1$, we get $A = -\frac{4}{15}$. Since the function $x \mapsto -3x^2 + px$ and $u \mapsto -u^2 + pu$ are concave, the function $(x, u) \mapsto H$ is concave. By Mangasarian's theorem, the conditions of the maximum principle are not only necessary but also sufficient.

3c (weight 10%)

The terminal condition in this case is either

$$x(\ln(2)) = -1 \text{ and } p(\ln(2)) \geq 0 \tag{6}$$

or

$$x(\ln(2)) > -1 \text{ and } p(\ln(2)) = 0. \tag{7}$$

The same derivation as in the previous question leads us to (5) and we have to determine A. If $x(\ln(2)) = -1$, it follows the previous question that $A = -\frac{4}{15}$ and

$$p(\ln(2)) = -\frac{10}{15}$$

so that (6) does not hold. If $p(\ln(2)) = 0$, we obtain $A = 0$ and $p(t) = x(t) = 0$. Hence, (7) is satisfied, as $x(\ln(2)) = 0 > -1$, and

$$x^*(t) = u^*(t) = 0$$

is a solution which fullfills the conditions of the maximum principle. Since the function $(x, u) \mapsto H$ is concave, we know, by Mangasarian's theorem, that this pair solves also the original optimal control problem.