UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2440 — Ordinary Differential Equations and Optimal Control Theory

Day of examination: Tuesday 14. June 2011

Solution proposal

Problem 1

1a

The equilibrium points satisfy

$$
\begin{aligned}
4y^3 - 4xy &= 0,\\
2y^2 - 2x^3 &= 0.
\end{aligned}
$$

The first equation gives $4y(y^2-x)=0$ and either $y=0$ or $x=y^2$. If $y=0$, the second equation yields $x=0$. If $x=y^2$, then the second equation gives $x(1-x^2)=0$, that is, $x=0$ or $x=\pm1$. If $x=-1$, then $y^2=-1$ and there does not exist an equilibrium point in this case. If $x=1$, we get $y=\pm$. In conclusion, there are 3 equilibrium points: $(0,0)$, $(1,1)$, $(1,-1)$.

We linearize around $(1,1)$. We compute the Jacobian:

$$
J(x,y) = \begin{pmatrix} -4y & 12y^2-4x \\ -6x^2 & 4y \end{pmatrix}.
$$

For $x=y=1$, we get

$$
J(1,1) = \begin{pmatrix} -4 & 8 \\ -6 & 4 \end{pmatrix}.
$$

The eigenvalues satisfies $\lambda^2 + 32 = 0$ and therefore they are complex conjugate numbers with zero real value. In this case, we cannot conclude if the equilibrium of the originial system is an attractive or repulsive point.

1b

Let $M(x,y) = 2x^3 - 2y^2$ and $N(x,y) = 4y^3 - 4xy$. We can rewrite the ordinary differential equation as the form $M\,dx + N\,dy = 0$. We have $\frac{\partial M}{\partial y} = -4y = \frac{\partial N}{\partial x}$ and therefore the form is exact and there exists a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. Then,

$$
\frac{\partial F}{\partial x} = 2x^3 - 2y^2
$$

implies

$$F = \frac{1}{2}x^4 - 2y^2x + g(y)$$

for some unknown function $g(y)$. We differentiate this result with respect to y and obtain $-4yx + g'(y) = 4y^3 - 4xy$. Therefore, $g(y) = y^4 + C$ and the solution of the ordinary differential equation are implicitely given by

$$F = \frac{1}{2}x^4 - 2y^2x + y^4 + C$$

for some constant C. We can rewrite F as

$$F = (x - y^2)^2 + \frac{1}{2}(x^2 - 1)^2 + C - \frac{1}{2}$$

1c

Let us consider a solution $(x(t), y(t))$. We have $\dot{x} = M(x, y)$ and $\dot{y} = -M(x, y)$. Hence,

$$\frac{d}{dt}F(x(t), y(t)) = \frac{\partial F}{\partial x}M + \frac{\partial F}{\partial y}(-N) = NM - MN = 0$$

and therefore $F(x(t), y(t))$ is a constant. Let us assume that the equilibrium $(1, 1)$ is a sink or spiral sink. Then, there exists a trajectory $(x(t), y(t))$ starting close but away from $(1, 1)$, that is, $(x(0), y(0)) \neq (1, 1)$ such that $\lim_{t\to\infty} x(t) = 1$ and $\lim_{t\to\infty} y(t) = 1$. We have

$$(x(t) - y^2(t))^2 + \frac{1}{2}(x^2(t) - 1)^2 = C$$

for some constant C. Since $(x(0), y(0)) \neq (1, 1)$, we have $C > 0$. By letting t tend to infinity, we obtain $0 = C$, which is a contradiction and therefore the equilibrium is not an attractive point.

Problem 2

We compute the eigenvalue of A. We have to solve

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(2 - \lambda)((3 - \lambda)(1 - \lambda) + 1) = 0$$

which yields $(2 - \lambda)(\lambda - 2)^2$ and $\lambda = 2$ is an eigenvalue with multiplicity 3. We compute the eigenvector space. We have that $u = [x, y, z]^t$ is an eigenvector if

$$(A - \lambda I)u = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This system of equations is equivalent to

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so that the eigenvector space is of dimension 2 for which

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

form a basis. We have to compute a generalized eigenvector v. It satisfies

$$(A - \lambda I)^{k-m+1} v = 0$$

where $k = 3$ is the multiplicity of the eigenvalue, $m = 2$ is the dimension of the eigenvector space. We get

$$(A - \lambda I)^{k-m+1} = (A - \lambda I)^2 = 0$$

and we choose $v = [0, 1, 0]^t$, which is linearly independent of u_1 and u_2. We set $v_2 = v$ and

$$v_1 = (A - \lambda I) v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The vectors v_1, v_2 form a chain. The general solution of the ordinary differential equation is given by

$$X(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + C_3 e^{2t} \left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

for any constants C_1, C_2, C_3. For the initial value $[0, 1, 0]$, we have to solve

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which gives $C_1 = C_2 = 0$, $C_3 = 1$ and the solution is

$$X(t) = e^{2t} \left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right).$$

Problem 3

3a

Let $H = (x^2 - u^2) + pu$. The optimal pair (x^*, u^*) satisfies

$$H(t, x^*(t), u^*(t), p(t)) = \max_{u \in [0,1]} H(t, x^*(t), u, p(t))$$

and

$$\dot{p} = -\frac{\partial H}{\partial x} = -2x$$

and

$$p(\pi) = 0.$$

Since $\dot{x} = u$ and $u \in [0,1]$, x is an increasing function and therefore $x(t) \geq x(0) = 1$ for $\in [0,\pi]$. Thus $\dot{p} \leq -2$ and p is strictly decreasing.

3b

We have $\frac{\partial H}{\partial u} = -2u + p = 0$ if $u = \frac{p}{2}$. Since the function $u \mapsto H$ is concave, the maximum is attained for $u^* = \frac{p}{2}$. This value of u^* belongs to $[0,1]$ when $p \in [0,2]$. For $p \notin [0,2]$, the maximum is attained at the boundaries of the interval $[0,1]$ and we have to compare the values of H for $u = 0$ and $u = 1$. We have

$$H(1) - H(0) = x^2 - 1 + p - x^2 = p - 1.$$

Hence, for $p < 0$, $H(1) < H(0)$ and the maximum is attained for $u^* = 0$. For $p > 2$, $H(1) > H(0)$ and the maximum is attained for $u^* = 1$.

3c

We have $p(\pi) = 0$. By continuity of p, there exists a $t_* < \pi$ such that $p(t) \in [-2,2]$ for $t \in [t_*,\pi]$. Since, the function p is strictly decreasing, we must have $p(t) \in [0,2]$.

For $p \in [0,2]$, $u^* = \frac{p}{2}$ and we have to solve the system of ordinary differential equations

$$\dot{x} = \frac{p}{2}$$
$$\dot{p} = -2x.$$

We differentiate the second equation and plug in the first one. We obtain

$$\ddot{p} + p = 0.$$

The general solution is $p(t) = A\cos(t) + B\sin(t)$. Since $p(\pi) = 0$, we get $p(t) = B\sin(t)$. Then,

$$x(t) = -\frac{\dot{p}}{2} = -\frac{B}{2}\cos(t)$$

and $x(\pi) = \bar{x}$ gives $\bar{x} = -\frac{B}{2}\cos(\pi)$, that is, $B = 2\bar{x}$. Finally, we obtain

$$x(t) = -\bar{x}\cos(t)$$
$$p(t) = 2\bar{x}\sin(t).$$

3d

Let us assume that $t_* = 0$. Then we must have $x(0) = 1 = -\bar{x}\cos(0)$. It gives $\bar{x} = -1$ and $p(t) = -2\sin(t)$. However, this last result contradicts the fact that $p(t) \in [0,2]$ for $t \in [t_*,\pi]$.

We have $p(t_*) = 2$ if and only if $2\bar{x}\sin(t_*) = 2$, that is, $\bar{x}\sin(t_*) = 1$

3e

Since p is strictly decreasing, we have $p(t) > 2$ for $t \in [0, t_*)$. Then, $u^* = 1$ and we have to solve

$$\dot{x} = 1$$
$$\dot{p} = -2x.$$

It gives

$$x(t) = t + 1,$$

as $x(0) = 1$, and $\dot{p} = -2x = -2t - 2$ implies

$$p(t) = -t^2 - 2t + \bar{p}.$$

By continuity of the functions p and x at t_*, we get

$$t_* + 1 = -\bar{x}\cos(t_*)$$
$$2\bar{x}\sin(t_*) = 2 = -t_*^2 - 2t_* + \bar{p}$$

Thus, $\bar{x} = \frac{1}{\sin(t_*)}$ and the first equation above gives

$$t_* + 1 = -\frac{1}{\tan(t_*)}.$$

We get $\bar{p} = 2 + t_*^2 + 2t_*$. The optimal control u^* is given by

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t_*], \\ \frac{\sin(t)}{\sin(t_*)} & \text{if } t \in [t_*, \pi]. \end{cases}$$