PROBLEM

A function F is given by

$$F(x,y) = x^2(1+y^2)$$

(**a**) Show that F is convex in the region

$$R = \{(x,y) \,:\, |y| \le \frac{1}{\sqrt{3}}\}$$

(**b**) Show that the variation problem

$$(1) \qquad \begin{aligned} &\min \int_0^1 x^2(1+\dot{x}^2)\,\mathrm{d}t, \\ &x(0) = x(1) = 1 \end{aligned}$$

has Euler equation

$$(*) \qquad x(x\ddot{x} + \dot{x}^2 - 1) = 0$$

Explain that the solutions of $(*)$ may be obtained as solutions of the equation

$$F(x,\dot{x}) - \dot{x}\frac{\partial F}{\partial \dot{x}}(x,\dot{x}) = c$$

where c is an arbitrary constant.

(**c**) Find the only possible solution x^* of the variation problem. Let

$$V = \{x \,:\, x \text{ is a } C^2 \text{ function such that } x(0) = x(1) = 1 \text{ and } |\dot{x}(t)| \le \frac{1}{\sqrt{3}} \text{ for all } t \in [0,1]\}$$

Show that $x^* \in V$. Explain that x^* minimizes the integral in (1) among all functions in V.

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