MAT 2440 Solutions

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Problem 1.

We observe that

$$A = 2I + N,\ N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ N^3 = 0$$

Since $IN = NI$, the matrix exponential is

$$\begin{aligned} e^{tA} &= e^{t(2I+N)} = e^{2t}Ie^{tN} = e^{2t}(I + tN + \frac{1}{2}t^2N^2) \\ &= e^{2t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Consequently, the solution of the initial value problem is

$$\mathbf{x}(t) = e^{2tA}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = e^{2t}\begin{bmatrix} t + \frac{1}{2}t^2 \\ 1 + t \\ 1 \end{bmatrix}$$

An alternative is to use that the eigenvalues are all equal to 2 (A is upper triangular so the eigenvalues are the entries on the main diagonal). The eigenspace is seen to be one-dimensional, spanned by $\mathbf{v}_1 = [1, 0, 0]^T$. Hence two independent generalized eigenvectors are needed. First $(A - 2I)^3 = 0$, so that any vector $\mathbf{v}_3 \neq \mathbf{0}$ that is linearly independent of $\mathbf{v}_1$ may be tried. We take $\mathbf{v}_3 = [0, 0, 1]^T$. Then

$$(A - 2I)\mathbf{v_3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{v_2}$$

Finally,

$$(A - 2I)\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v_1}$$

Using the above generalized eigenvectors we find the general solution

$$\mathbf{x}(t) = ae^{2t}\mathbf{v_1} + be^{2t}(t\mathbf{v_1} + \mathbf{v_2}) + ce^{2t}(\frac{1}{2}t^2\mathbf{v_1} + t\mathbf{v_2} + \mathbf{v_3})$$

Then

$$\mathbf{x}(0) = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = [a, b, c]^T = [0, 1, 1]^T \Leftrightarrow a = 0, b = c = 1.$$

Hence we readily derive the same solution as above.

Problem 2
(**a**) The critical points occur for
$y = \sqrt{x}/\varepsilon$ (if $\varepsilon \neq 0$)
and
$2x + y/\sqrt{x} - 1 = 0$
This leads to $2x + 1/\varepsilon - 1 = 0$, and the only critical point is

$$x = \frac{1}{2}(1 - \frac{1}{\varepsilon}),\ y = \frac{1}{\varepsilon\sqrt{2}}(1 - \frac{1}{\varepsilon})^{\frac{1}{2}}.$$

Since $x > 0$ we must have $\frac{1}{\varepsilon} < 1$, that is $\varepsilon > 1$ or $\varepsilon < 0$.
(**b**) Here

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2x + \frac{y}{\sqrt{x}} - 1}{2\varepsilon y - 2\sqrt{x}},$$

or

$$(2x + \frac{y}{\sqrt{x}} - 1)\,\mathrm{d}x + (2\sqrt{x} - 2\varepsilon y)\,\mathrm{d}y = 0,$$

which is of the type $Pdx + Qdy = 0$. Such differential forms are exact if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. In the present case this holds true for all ε, both expressions being equal to $\frac{1}{\sqrt{x}}$. Hence there is a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$. Integration of P wrt. x yields

$$F(x, y) = x^2 + 2y\sqrt{x} - x + A(y),$$

and then

$$\frac{\partial F}{\partial y} = 2\sqrt{x} + A'(y) = 2\sqrt{x} - 2\varepsilon y,\ A'(y) = -2\varepsilon y.$$

Thus

$$A(y) = -\varepsilon y^2 + k.$$

Now the solutions of our differential equation are given by $F(x, y(x)) = C$, or

$$(*) \qquad x^2 - \varepsilon y^2 + 2y\sqrt{x} - x = C$$

(**c**) Let $\varepsilon = -1$. Then the critical point is $(1, -1)$. We complete the squares in $(*)$ and derive

$$\left(\sqrt{-\varepsilon}y + \sqrt{x}\,\right)^2 + (x - \frac{1}{2}(1 - \frac{1}{\varepsilon}))^2 = K \qquad (K \text{ a constant})$$

With $\varepsilon = -1$ this becomes

$$(**) \qquad \left(y + \sqrt{x}\,\right)^2 + (x-1)^2 = K$$

If the critical point were repulsive, then $x \to \infty$ or $y \to \pm\infty$ as $t \to \infty$. This clearly contradicts $(**)$ since its right hand side is constant. Similarly, it is impossible to have $x \to 1$ and $y \to -1$ as $t \to \infty$ (the constant solution $x = 1, y = -1$ cannot occur since the solution curves are supposed not to pass through the critical point.) Thus the critical point is not a sink. (We conclude that the critical point is either a stable center or an asymptotic stable spiral point.)

(**d**) We let $\varepsilon = 1$ in (1):

$$(i) \qquad \dot{x} = 2y - 2\sqrt{x}$$

$$(ii) \qquad \dot{y} = 2x + \frac{y}{\sqrt{x}} - 1$$

From (i)

$$y = \frac{1}{2}(\dot{x} + 2\sqrt{x}) = \frac{1}{2}\dot{x} + \sqrt{x},\ \dot{y} = \frac{1}{2}\ddot{x} + \frac{1}{2}\frac{\dot{x}}{\sqrt{x}}$$

We combine this with (ii):

$$\frac{1}{2}\ddot{x} + \frac{\dot{x}}{2\sqrt{x}} = 2x + \frac{\frac{1}{2}\dot{x} + \sqrt{x}}{\sqrt{x}} - 1 = 2x + \frac{\dot{x}}{2\sqrt{x}}$$

or

$$\ddot{x} - 4x = 0$$

Hence

$$x(t) = Ae^{2t} + Be^{-2t}$$

and

$$y(t) = \frac{1}{2}\dot{x} + \sqrt{x} = Ae^{2t} - Be^{-2t} + \sqrt{Ae^{2t} + Be^{-2t}}$$

Problem 3

We will solve

$$\max \int_0^1 (x - x^2 - u^2)\,\mathrm{d}t,\ \dot{x} = -2\sqrt{x} - 2u,\ x(0) = 1,\ x(1) = 0.$$

(**a**) The Hamiltonian for this (normal) problem is

$$H = H(t, x, u, p) = x - x^2 - u^2 + 2p(-\sqrt{x} - u)$$

If $x = x^*, u = u^*$ form an optimal pair, then by the Maximum Principle there is a continuous and piecewise C^1–function p such that

$$\frac{\partial H}{\partial x} = -\dot{p}.$$

That is,

$$\dot{p} = 2x + px^{-\frac{1}{2}} - 1.$$

Moreover, $u = u^*$ must maximize $H(t, x^*(t), u, p(t))$ for each $t \in [0, 1]$. Hence (as the range of $u = u(t)$ is all of $\mathbb{R}$) we must have $\frac{\partial H}{\partial u} = 0$ or

$$-2u - 2p = 0, \ u = -p$$

Since $\frac{\partial^2 H}{\partial u^2} = -2 < 0$, this yields a maximum. We combine this with the relation $\dot{x} = -2\sqrt{x} - 2u$ and obtain the system

$$\text{(I)} \qquad \begin{cases} \dot{x} = 2p - 2\sqrt{x} \\ \dot{p} = 2x + \frac{p}{\sqrt{x}} - 1\,, \quad x > 0, \end{cases}$$

(**b**) The system (I) is (1) of Problem 2 with $y = p$ and $\varepsilon = 1$. The solution for $x = x^*$ of 2(d) was

$$x(t) = Ae^{2t} + Be^{-2t}$$

Here

$$x(0) = A + B = 1$$

and

$$x(1) = Ae^2 + Be^{-2} = 0$$

which give

$$B = -e^4 A, \ A(1 - e^4) = 1,$$

$$A = \frac{1}{1 - e^4}, \ B = -\frac{e^4}{1 - e^4} = \frac{e^4}{e^4 - 1}$$

Hence

$$x^*(t) = \frac{1}{1 - e^4}(e^{2t} - e^{4-2t}) = \frac{e^2}{e^4 - 1}(e^{2-2t} - e^{2t-2})$$

Then

$$\begin{aligned} u^*(t) &= -p(t) = -\frac{1}{2}\dot{x}(t) - \sqrt{x(t)} \\ &= \frac{1}{e^4 - 1}(e^{2t} + e^{4-2t}) - \frac{1}{\sqrt{e^4 - 1}}\sqrt{e^{4-2t} - e^{2t}} \end{aligned}$$

This is the only possible candidate of an optimal pair.

(**c**) Let $R(t) = \{(x,u) \in \mathbb{R}^2 \,:\, 4x^{3/2} \geq p(t)\}$

We proceed to show, for each t, $H(t,x,u,p(t))$ is concave with respect to (x,u) in the region $R(t)$ by using the 2nd derivative test. Let $t \in [0,1]$ and put $p = p(t)$. Here

$$\frac{\partial^2 H}{\partial u^2} = -2 < 0$$

and

$$\frac{\partial^2 H}{\partial x^2} = -2 + \frac{1}{2} p x^{-3/2} \leq 0 \Leftrightarrow p x^{-3/2} \leq 4 \Leftrightarrow p \leq 4x^{3/2}$$

Finally,

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial u^2} - \frac{\partial^2 H}{\partial x \partial u} = -2\frac{\partial^2 H}{\partial x^2} \geq 0 \Longleftrightarrow p x^{-3/2} \leq 4$$

This shows the convexity statement.

(**d**) Assume that the solution (x^*, u^*) from (b) belongs to W this is the case if and only if $4(x^*)^{3/2} < p(t)$ for all $t \in [0,1]$ and may be proved as indicated below.

Now the function $(x,u) \mapsto H(t,x,u,p(t))$, $W \to \mathbb{R}$ is concave for each $t \in [0,1]$. By Mangasarian's Theorem the pair (x^*, u^*) from (b) is optimal among the elements of W: The proof of Mangasarian's Theorem works equally well if concavity of H holds only in an open, convex subset of the $xu-$plane. In the present case the region

$$\{(x,u) \,:\, x^{3/2} > \frac{1}{4}p\}$$

is open and convex. (If $p > 0$ this is the half plane $\{(x,u) \,:\, x > (\frac{1}{4}p)^{2/3}\}$, and if $p < 0$ it is the set $\mathbb{R}^2$.) Convexity is needed where the "Gradient Inequality" is used in the proof.

For the sake of completeness we finally prove that $p(t) < 4x^{3/2}$, for all $t \in [0,1]$, and hence the solution (x^*, u^*) from (b) belongs to W:

$$p(t) = \Big(\frac{e^{4-2t} - e^{2t}}{e^4 - 1}\Big)^{\frac{1}{2}} - \Big(\frac{e^{4-2t} - e^{2t}}{e^4 - 1}\Big) < \frac{4}{(e^4-1)^{3/2}}(e^{4-2t} - e^{2t})^{3/2} = 4x^{3/2}$$

$$\Updownarrow$$

$$(e^{4-2t} - e^{2t})^{\frac{1}{2}}(e^4 - 1) - (e^{4-2t} + e^{2t})(e^4 - 1) < 4(e^{4-2t} - e^{2t})^{3/2}$$

We let $y = e^{2t}, k = e^4$. Then the last inequality is equivalent to:

$$(ky^{-1} - y)^{1/2}(k-1) - 4(k^{-1} - y)^{3/2} < (ky^{-1} + y)(k-1)$$

or

$$(k-1) - 4(ky^{-1} - y) < \frac{ky^{-1} + y}{(ky^{-1} - y)^{1/2}}(k-1)$$

Let

$$\phi(y) = (k-1) - 4(ky^{-1} - y),$$
$$\psi(y) = \frac{ky^{-1} + y}{(ky^{-1} - y)^{1/2}}(k-1).$$

Then

$$\phi'(y) = -4(-ky^{-2} - 1) = 4(k^{-2} + 1) > 0.$$

Hence ϕ increases and its maximum is

$$\phi_{\max} = \phi(e^2) = k - 1.$$

Therefore it suffices to prove that

$$\psi_{\min} > \phi_{\max} = k - 1.$$

Now

$$\psi'(y) = \frac{k-1}{ky^{-1} - y}[(1-ky^{-2})(ky^{-1}-y)^{1/2} + \frac{1}{2}(ky^{-1}+y)(ky^{-1}-y)^{-1/2}(ky^{-2}+1)]$$
$$= \frac{(k-1)y}{(ky^{-1} - y)^{3/2}}[\frac{1}{2}(ky^{-2} + 1)^2 - (1 - ky^{-2})^2]$$
$$= \frac{k-1}{(ky^{-1} - y)^{3/2}}[-\frac{1}{2}k^2y^{-4} + 3ky^{-2} - \frac{1}{2}]$$

Let $f(y) = -k^2y^{-4} + 6ky^{-2} - 1$. Then

$$\psi'(y) = \frac{\frac{1}{2}(k-1)y}{(ky^{-1} - y)^{3/2}}f(y) = r(y)f(y),$$

where $r(y) > 0$. Thus the sign and zeros of ψ' are completely determined by f. Next we put $u = y^{-2}, y = u^{-1/2}$, $f(y) = f(u^{-1/2}) = g(u)$. Then

$$g(u) = -k^2u^2 + 6ku - 1 = 0$$

if and only if

$$u = \frac{1}{2k^2}[6k \pm \sqrt{32k^2}] = \frac{1}{k}[3 \pm 2\sqrt{2}]$$

Since u is in $[1/k, 1]$ we must use the $+$ sign in front of the square root. Here $g(1/k) = 4 > 0$ and $g(1) = -k^2+6k-1 = -e^8+6e^4-1 < 0$. Hence $\psi'(y) > 0$ for $y \in (e^2/(3+2\sqrt{2})^{1/2}, e^2)$ and $\psi'(y) < 0$ for $y \in (1, e^2/(3+2\sqrt{2})^{1/2})$. Let $a = (3+2\sqrt{2})^{1/2}$. Then

$$\psi_{\min} = \psi(e^2/a) = \frac{e^2a + e^2/a}{(e^2a - e^2a^{-1})^{1/2}}(e^4 - 1)$$
$$= \frac{e(a + a^{-1})}{(a - a^{-1})^{1/2}}(e^4 - 1) = \frac{e(2.82...)}{1.41...}(e^4 - 1) > e^4 - 1 = \phi_{\max}$$

This shows that $p(t) < 4x^{3/2}$, for all $t \in [0, 1]$.

THE END