OBLIGATORY PROBLEMS IN MAT 2440

Deadline April 11, 2013, 2:30 p.m.

All the problems can be solved without the aid of a computer or a calculator, however you may also use any computer system (like Matlab, Mathematica, Maple, or the numpy package of Python). Whenever doing so, you should document the commands you are using and explain in detail how you apply the results.

PROBLEM 1

Let

$$P(x, y) = \frac{2y^2}{x(y^2 - x^2)}$$

and

$$Q(x, y) = \frac{-2x^2}{y(y^2 - x^2)}$$

whenever $x > 0, y > 0$, and $y \neq x$.

Solve the differential equation

$$Pdx + Qdy = 0$$

(**a**) as an exact equation

(**b**) as a homogeneous equation.

PROBLEM 2

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Find a general solution of the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$

(**a**) by using the method of elimination (write the equation as a system of linear first-order equations), (**b**) by the eigenvalue method.

PROBLEM 3

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Find a general solution of the equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

PROBLEM 4

Let

$$F(t, x, \dot{x}) = \frac{1}{2}(1 - t^2)\dot{x}^2 - 3x^2 + x\dot{x} + tx + \frac{1}{2}t^2\dot{x}$$

We consider the following problem

$$(*) \qquad \max_x \int_3^5 F(t, x, \dot{x})\, dt, \quad x(3) = 0,\ x(5) = \frac{37}{4}\ln\frac{4}{3} - \frac{69}{26}$$

(**a**) Show that the Euler equation of the problem can be written

$$(1) \qquad (1 - t^2)\ddot{x} - 2t\dot{x} + 6x = 0$$

(**b**) Verify that the equation in (1) has a polynomial solution

$$x_1(t) = 1 - 3t^2$$

We proceed to find a linearly independent solution x_2 such that

$$x_2(t) = v(t)x_1(t),$$

for some C^2-function v on the interval $I = [3, 5]$.

(**c**) Prove that v must satisfy the differential equation

$$(2) \qquad \ddot{v}(t) + \frac{2t(9t^2-7)}{(1-t^2)(1-3t^2)}\dot{v}(t) = 0 \quad (\text{for all } t \in I)$$

Write

$$q(t) = \frac{2t(9t^2-7)}{(1-t^2)(1-3t^2)}$$

and find numbers a and b such that

$$q(t) = 2t(\frac{a}{1-t^2} + \frac{b}{1-3t^2})$$

(**d**) Show that (2) has a solution for $\dot{v}$ given by

$$(3) \qquad \dot{v}(t) = \frac{1}{(1-t^2)(1-3t^2)^2} \qquad (\text{for all } t \in I)$$

(**e**) Find the unique solution x^* of of the Euler-equation (1) that satisfies the given endpoint conditions.

HINT: $\dot{v}$ may be written

$$\dot{v}(t) = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+\sqrt{3}t} + \frac{D}{1-\sqrt{3}t} + \frac{Et+F}{(1+\sqrt{3}t)^2} + \frac{Gt+H}{(1-\sqrt{3}t)^2}$$

(**f**) Decide if the solution x^* in (e) solves the maximum problem in $(*)$. Justify your answer.

THE END