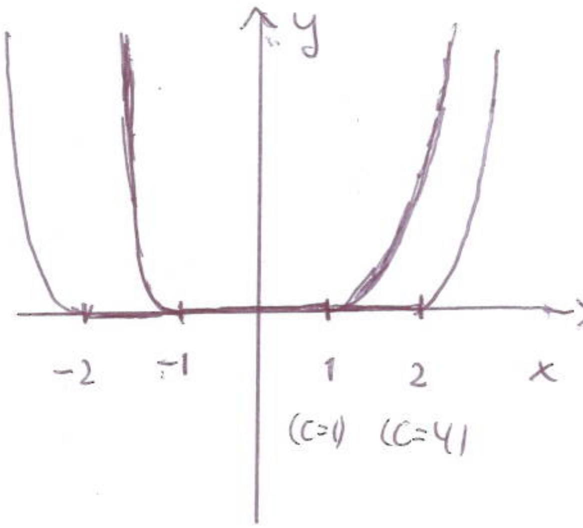
1.3 (32) Let $c > 0$ and let




(*) $y(x) = \begin{cases} 0, & \text{if } x^2 \le c \\ (x^2-c)^2, & \text{if } x^2 > c \end{cases}$

We claim that

(**) $y'(x) = 4x\sqrt{y}$, for all $x$.

$x^2 < c$: Then $y(x) = 0$, hence $y'(x) = 0 = 4x\sqrt{y}$.

$x^2 > c$: $y(x) = (x^2-c)^2$,

hence

$$y'(x) = 4x(x^2-c)$$

and $4x\sqrt{y} = 4x(x^2-c) = y'(x)$, since $x^2 > c$.

$x^2 = c$: Let $h < 0$. Then at $x = c^{1/2}$ $(x^2 = c)$,

$$\frac{y(c^{1/2}+h) - y(c^{1/2})}{h} = \frac{0-0}{h} = 0,$$

so that $y$ has a left derivative at $x = c$:

$$\lim_{h\to 0^-} \frac{y(c^{1/2}+h)-y(c^{1/2})}{h} = \underline{0}$$

Let $h > 0$. Then

$$\frac{y(c^{1/2}+h)-y(c^{1/2})}{h} = \frac{(c^{1/2}+h)^2 - 0}{h} \xrightarrow[h\to 0^+]{} \underline{0}.$$

We have shown that $y'(c^{1/2})$ exists and is equal to $0$.
Hence $4c^{1/2}\sqrt{y(c^{1/2})} = 0 = y'(c^{1/2})$.

This proves (**):

Let $(a, b)$ be given. We shall find solutions of the initial value problem

$$(***) \quad y'(x) = 4\sqrt{y} \quad , \quad y(a) = b$$

Clearly, there is no solutions if $b < 0$ since $y \geq 0$ here ($\sqrt{y}$ is only defined if $y \geq 0$).

If $b \geq 0$ and $a^2 > b^{1/2}$, we can always find $c$ such that

$$(a^2 - c)^2 = b \quad , \quad \text{i.e. } a^2 - c = b^{1/2},$$

just take $c = a^2 - b^{1/2}$.

If $a^2 \leq b^{1/2}$ $(b \geq 0)$, then there is no solution of $(a^2 - c)^2 = b$ , $a^2 - c = b^{1/2}$, since $c > 0$ (by what we have assumed).

To summarize:

$$y(x) = \begin{cases} 0, & \text{if } x^2 \leq a^2 - b^{1/2} = c \\ (x^2 - (a^2 - b^{1/2}))^2, & \text{if } x^2 > a^2 - b^{1/2} = c \end{cases}$$

is a solution of (***) if $a^2 > b^{1/2}$.

If $a > 0$ and $0 < k < c = a^2 - b^{1/2}$, the function

$$(****)\quad y(x) = \begin{cases} (x^2 - (a^2 - b^{1/2}))^2, & \text{if } x^2 > a^2 - b^{1/2} = c \\ 0, & \text{if } -k^{1/2} < x < c^{1/2} \\ (x^2 - k)^2, & \text{if } x < -k^{1/2} \end{cases}$$

is also a solution of the initial value problem (***), defined for all $x$. Since this applies to all $k$ with $0 < k < c$, there are infinitely many such solutions. Similar solutions exist if $a < 0$.

If $b = 0$, then there is a solution

$$y = (x^2 - a^2)^2$$

defined for all $x$ and, in addition, infinitely many solutions given by (****) and $c = a^2 > 0$.

If $a = 0$ and $b = 0$, then $y = x^4$ is a solution.

Other solutions are $y(x) = \begin{cases} x^4, & x \geq 0 \\ 0, & -k \leq x < 0 \\ (x^2-k)^2, & x < -k^{1/2} \end{cases}$ ; where $k > 0$. Hence infinitely many solutions

<u>By Theorem 1 p. 24 (§1.3)</u>: In the region

$$R = \{(x,y) : y > 0\}$$

the function $f(x,y) = 4x\sqrt{y}$ has a continuous partial derivative $(D_y f)(x,y) = \frac{2x}{\sqrt{y}}$. Hence the initial value problem (***) has a <u>unique</u> solution on some interval $I_a$ containing $a$, namely $y = (x^2 - c)^2$, provided that $b > 0$ (it is unique on $I_a$, but on all of $\mathbb{R}$ we have seen that there can be infinitely many solutions).

The unique solution given locally by Theorem 1 is obtained here by separation of variables in

$$(**) \quad y' = 4x\sqrt{y} \qquad (y > 0)$$

Hence $\dfrac{y'}{2\sqrt{y}} = 2$

which yields

$$y(x) = (x^2 - c)^2 \quad , \quad c \text{ an arbitrary constant.}$$

This means that all solutions of the initial value problem $(***)$ must be given by the formula $y(x) = (x^2 - c)^2$ in an open interval that contains $x = a$ (assuming $b > 0$).