UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT 2440 — Differential equations and Optimal Control Theory

Day of examination: Friday June 8, 2016

Examination hours: 09:00 – 13:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

SOLUTIONS:

Problem 1 (Weight 15 %)

Show that the differential equation

$$(1) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{4y^3 - y}$$

yields an exact differential form. Solve the equation (1) implicitly. Show that the solutions are given by the equation

$$(2) \qquad x^2 + y^2 - 2y^4 = C,$$

where C is a constant.

Solution:
(1) can be written as

$$(*) \quad x\,\mathrm{d}x + (y - 4y^3)\,\mathrm{d}y = 0$$

We let $P(x,y) = x,\ Q(x,y) = y - 4y^3$. Then

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}.$$

Hence $(*)$ is exact. Consequently, there is a "potential" function ϕ such that

$$\frac{\partial \phi}{\partial x} = P,\ \frac{\partial \phi}{\partial y} = Q.$$

Integrating the first of the identities with respect to x yields

$$\phi(x,y) = \frac{1}{2}x^2 + A(y)$$

(Continued on page 2.)

Then $\frac{\partial\phi}{\partial y} = A'(y) = y - 4y^3$. Hence $A(y) = \frac{1}{2}y^2 - y^4 + k$. One such ϕ is $\phi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - y^4$. Therefore, the solutions of (1) are given by

$$x^2 + y^2 - 2y^4 = C.$$

Problem 2 (Weight 45 %)

We will study the autonomous system of differential equations:

$$(3) \qquad \begin{cases} \dot{x} = 4y^3 - y \\ \dot{y} = x \end{cases}$$

(a) Find the critical points of the system.
Linearize (3) at the points $(0, \frac{1}{2})$ and $(0,0)$. Explain that the system is almost linear at both points.

(b) Solve the linear system that you obtained at $(0,0)$.

(c) Determine the type of the point $(0, \frac{1}{2})$ with regard to the the nonlinear system (3). What can you say at present about the type of $(0,0)$? Show that $(0,0)$ is no sink.

(d) Justify that $(0,0)$ is a center for the system (3).
Hint: Try polar coordinates.

Solution:
(a) : The critical points (also called equilibriums) are given by $4y^3 - y = 0$ and $x = 0$, equivalently: $x = 0$ and ($y = 0$ or $4y^2 = 1$). Hence they are exactly

$$(0,0), \ (0, \frac{1}{2}), \ (0, -\frac{1}{2}).$$

We linearize (3) at
(i) $(0, \frac{1}{2})$: Let $f(x,y) = 4y^3 - y, \ g(x,y) = x$. Then the Jacobian matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 12y^2 - 1 \\ 1 & 0 \end{bmatrix}, \text{ hence } A = J(0, \frac{1}{2}) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

We translate the critical point to the origin by $u = x, \ v = y - \frac{1}{2}$. Then the linearized system is

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2v \\ u \end{bmatrix}, \text{ or } \dot{u} = 2v, \dot{v} = u.$$

(ii) $(0,0)$: f and g being polynomial functions, we see directly from (3) that

$$(*) \quad \dot{x} = -y, \ \dot{y} = x$$

is the linearized system.
We notice that the polynomial functions f and g are continuously

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differentiable. Hence, if we change coordinates so that that the critical point is at the origin in the new (u, v)-system, we know from the course that the first order remainder term of Taylor's formula tends to zero faster than $\sqrt{u^2 + v^2}$. Moreover, the Jacobian matrices $J(0, \frac{1}{2})$ and $J(0, 0)$ are nonsingular. Since there are only finitely many critical points, they are all isolated. We conclude that the system (3) is almost linear at $(0, \frac{1}{2})$ and $(0, 0)$.
(**b**) : From (ii)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where the eigenvalues of

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

are $\pm i$. Eigenvectors $\begin{bmatrix} a \\ b \end{bmatrix}$ corresponding to the eigenvalue $-i$ are given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -i \begin{bmatrix} a \\ b \end{bmatrix},$$

or $ia = b$. Thus a complex eigenvector is $\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and a complex solution of the system $(*)$ is

$$e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t - i \sin t)(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$

This yields two real, linearly independent solutions that generate the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \quad (t \in \mathbb{R})$$

<u>Alternative:</u>
This can also be seen using elimination:

$$\begin{aligned} &\dot{x} = -y,\ \dot{y} = x \Rightarrow \ddot{x} = -\dot{y} = -x \Rightarrow \\ &\ddot{x} + x = 0 \Rightarrow \\ &x(t) = c_1 \cos t + c_2 \sin t, \\ &y(t) = -\dot{x}(t) = c_1 \sin t - c_2 \cos t \end{aligned}$$

(**c**) : The matrix $J(0, \frac{1}{2}) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ of the system at $(0, \frac{1}{2})$ has eigenvalues λ such that: $0 = \begin{vmatrix} \lambda & -2 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 2$, so that $\lambda = \pm\sqrt{2}$. They are real of opposite signs. Hence the point $(0, \frac{1}{2})$ is a <u>saddle point</u> both for the linear and the nonlinear system. Saddle points are unstable. The solution curves (trajectories) of (3) "look like" hyperbolas close to the critical point. The matrix $J(0, 0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ of the system at $(0, 0)$ has purely imaginary

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eigenvalues $\pm i$. Hence $(0,0)$ is either a center or a spiral point for the nonlinear system. It can be stable, unstable or asymptotically stable.

Suppose $(x(t), y(t))$ is a solution curve starting at a point $(x_0, y_0) \neq (0,0)$ close to the critical point $(0,0)$ and such that

$$||(x(t), y(t)) - (0,0)|| \to 0 \text{ as } t \to \infty.$$

Since $(x_0, y_0) \neq (0,0)$, the constant C of equation (2) $x^2 + y^2 - 2y^4 = C$ in Problem 1 must be nonzero (if $x^2 + y^2$ is small, say less than $1/\sqrt{2}$, then $x^2 + y^2 - 2y^4 > 0$, so $C > 0$). We have $x(t)^2 + y(t)^2 \xrightarrow[t\to\infty]{} 0$, hence $x(t)^2, y(t)^2$, and $y(t)^4$ all tend to 0 as $t \to \infty$. However, this implies the left side of (2) approaches zero, contradicting that the constant C is nonzero. We conclude that $(0,0)$ is no sink.

(**d**) : We use polar coordinates in the implicit solution formula (2):

$$x = r\cos\theta,\ y = r\sin\theta \text{ where } r = r(t), \text{ and } \theta = \theta(t) \text{ may depend on } t.$$

Using (2) we then find

$$(*) \qquad 2r^4\sin^4\theta - r^2 + C = 0$$

We notice that for points sufficiently close to $(0,0)$, we have $C > 0$ since y^4 becomes "much" smaller than $x^2 + y^2$ as $(x,y) \to (0,0)$. In fact, it suffices that $x^2 + y^2 < \frac{1}{2}$, that is, $r < \frac{1}{\sqrt{2}}$. We solve $(*)$ for r^2:

$$r^2 = \begin{cases} \frac{1}{4\sin^4\theta}[1 - \sqrt{1 - 8C\sin^4\theta}], & \text{if } \sin\theta = 0, \\ C, & \text{if } \sin\theta = 0. \end{cases}$$

We must use the minus sign in front of the square root, as a plus sign yields $r^2 \to \infty$ as $\theta \to 0$, contradicting $(*)$. On the other hand, the minus sign yields a "$\frac{0}{0}$" expression that is seen to approach C as $\sin\theta \to 0$, by l'Hôpital's rule, in agreement with $(*)$. Thus each $\sin^2\theta$ yields a unique r^2, hence a unique r (since $r \geq 0$). Acordingly, $\sin^2\theta$ being a periodic function of θ, we see from $(*)$ that solution curves starting sufficiently close to $(0,0)$ (say, for which $r < \frac{1}{\sqrt{2}}$) are periodic functions of θ. Hence they are closed. This implies that $(0,0)$ is a center. (Since $\sin^4(-\theta) = \sin^4\theta$ and $\sin(\pi - \theta) = \sin\theta$ we also see that the curves are symmetric about both coordinate axis in the xy-system.)

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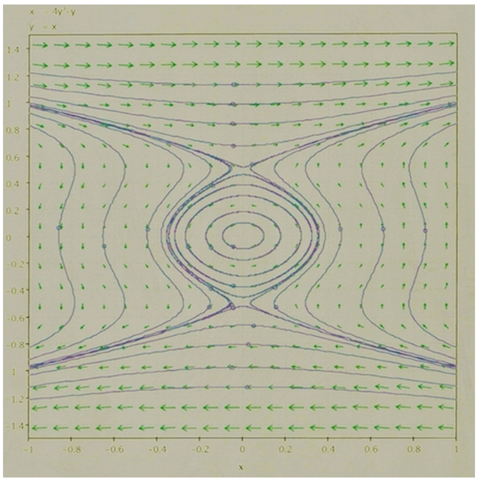
Phase plane portrait with a few trajectories of system (3)**:**
(A sketch of solution curves was not required.)



Problem 3 (Weight 40 %)

Assume that (x^*, u^*) is an optimal pair of the control problem:

$$\max \int_0^2 [x(t) + 2u(t)]e^{-t}\,\mathrm{d}t, \quad \dot{x} = 2x - \frac{1}{2}u,\ x(0) = 0,\ x(2) \text{ is free},$$
$$u(t) \in [0, 1] \text{ for all } t \in [0, 2].$$

(**a**) : Show that the adjoint function of the problem (as given in the Maximum Principle) is

$$p(t) = e^{2-2t} - e^{-t}, \qquad \text{for all } t \text{ in } [0, 2].$$

(**b**) Explain that $u^*(t) = 1$ or $u^*(t) = 0$ for all t in $[0, 2]$. Find (x^*, u^*). Decide if this really is an optimal pair.

(**c**) Find an optimal pair of the following normal control problem, if an optimal pair exists:

$$\max \int_0^2 [x(t) + 2u(t)]e^{-t}\,\mathrm{d}t, \quad \dot{x} = 2x - \frac{1}{2}u,\ x(0) = 0,\ x(2) \geq e^4 + \frac{1}{4},$$
$$u(t) \in [0, 1] \text{ for all } t \in [0, 2].$$

Hint: Verify that $\dot{x} \leq 2x$.

Solution:
(**a**) : The Hamiltonian of the problem is

$$H(t, x, u, p) = (x + 2u)e^{-t} + p(2x - \frac{1}{2}u)$$

By the Maximum Principle there is an "adjoint function" p that is continuous, piecewise C^1, and is given by the differential equation

$$\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t)) = -\dot{p}(t),$$

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except at the discontinuities of u^*. The above yields $e^{-t}+2p = -\dot{p}$, or $\dot{p}+2p = -e^{-t}$, $\frac{\mathrm{d}}{\mathrm{d}t}(pe^{2t}) = -e^{t}$, which has the general solution $p(t) = ae^{-2t} - e^{-t}$. Here $p(t) = 0$ since $x(2)$ is free. Hence

$$p(t) = e^{2-2t} - e^{-t}.$$

(**b**) :
By the Maximum Principle again, $u = u^*(t)$ must maximize the function h_t, where

$$h_t(u) = H(t, x^*(t), u, p(t)) = x^*(t)(e^{-t} + 2p(t)) + u(2e^{-t} - \frac{1}{2}p(t))$$

for each $t \in [0, 2]$. Here H is linear in u (even in (x, u)), so the maximum must be attained at an endpoint $u = 0$ or $u = 1$. Consequently, $u^*(t) = 0$ or $u^*(t) = 1$. We have

$$u^*(t) = \begin{cases} 0, & \text{if } 2e^{-t} - \frac{1}{2}p(t) < 0, \text{ i.e. } p(t) > 4e^{-t}, \\ 1, & \text{if } p(t) < 4e^{-t}, \\ \text{any value in } [0,1], & \text{if } p(t) = 4e^{-t}, \\ \text{we let } u^*(t) = 0 \text{ in this case.} & \end{cases}$$

Since $p(t) = e^{2-2t} - e^{-t}$, we find

$$\begin{aligned} & p(t) < 4e^{-t} \Leftrightarrow e^{2-2t} - e^{-t} < 4e^{-t} \\ & \Leftrightarrow e^{2-2t} < 5e^{-t} \Leftrightarrow e^{2-t} < 5 \Leftrightarrow 2 - t < \ln 5 \\ & \Leftrightarrow t > 2 - \ln 5 \quad (\text{where } 2 - \ln 5 \in (0, 2)). \end{aligned}$$

Hence

$$u^*(t) = \begin{cases} 0, & \text{if } 0 \le t \le 2 - \ln 5 \\ 1, & \text{if } 2 - \ln 5 < t \le 2 \end{cases}$$

Notice that $x = x^*, u^*$ satisfy $\dot{x} = 2x - \frac{1}{2}u$. There are two cases to consider:
(**i**) $\underline{\mathbf{u = 0}}$ ($t \in [0, 2 - \ln 5]$) yields
$\dot{x} = 2x, \quad x(t) = ce^{2t}$. Since $x(0) = 0$, we find

$$x^*(t) = x(t) = 0 \quad (t \in [0, 2 - \ln 5]).$$

(**ii**) $\underline{\mathbf{u = 1}}$ ($t \in [2 - \ln 5, 2]$) implies
$\dot{x} - 2x = -\frac{1}{2}, \quad x(t) = Ke^{2t} + \frac{1}{4}$. Since $x = x^*$ is continuous, we find

$$\begin{aligned} 0 = x(2 - \ln 5) &= Ke^{2(2-\ln 5)} + \frac{1}{4} \\ &= Ke^4\frac{1}{25} + \frac{1}{4}, \\ K &= (-\frac{1}{4})e^{-4}25 = -\frac{25}{4}e^{-4}, \\ x^*(t) = x(t) &= -\frac{25}{4}e^{-4+2t} + \frac{1}{4} \end{aligned}$$

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Since $H(t, x, u, p(t))$ is a linear function of (x, u) for each fixed $t \in [0, 2]$, it is certainly concave (and convex). By Mangasarian's Theorem the pair (x^*, u^*) is optimal.

(**c**) :

Since this is the same problem as above, but with the new terminal condition $x(2) \geq e^4 + \frac{1}{4}$, we obtain the same Hamiltonian and the same solution for p (up to a constant): $p(t) = ae^{-2t} - e^{-t}$. Again

$$u^*(t) = \begin{cases} 0, & \text{if } 2e^{-t} - \frac{1}{2}p(t) < 0, \text{ i.e. } p(t) > 4e^{-t}, \\ 1, & \text{if } p(t) < 4e^{-t}, \\ \text{any value in } [0, 1], & \text{if } p(t) = 4e^{-t}, \text{we let } u^*(t) = 0 \text{ in this case,} \end{cases}$$

hence there are two cases for $x = x^*$. Pursuing this, we may eventually (after some work) obtain a contradiction. However, there is an easier solution, as indicated by the **Hint:**

The equation of state, $\dot{x} = 2x - \frac{1}{2}u$, where $u \in [0, 1]$, yields that

$$(i) \qquad \dot{x} \leq 2x.$$

Since $x(0) = 0$, this implies that x never can become positive, $x(t) \leq 0$ for all $t \in [0, 2]$. The following argument proves this:

We multpply the inequality $\dot{x}(t) - 2x(t) \leq 0$ by e^{-2t}. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)e^{-2t}) = e^{-2t}(\dot{x}(t) - 2x(t)) \leq 0.$$

Thus the function $x(t)e^{-2t}$ is decreasing. Since $x(0) = 0$, $x(t)e^{-2t} = 0$ at $t = 0$. It follows that $x(t)e^{-2t} \leq 0$, and hence $x(t) \leq 0$, for all $t \geq 0$.

Alternative: On intervals I where $x(t) > 0$, (i) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln x(t) = \frac{\dot{x}(t)}{x(t)} \leq 2, \text{ hence } \ln x(t) \leq 2t + k.$$

Therefore, $0 < x(t) \leq Ke^{2t}$ $(t \in I)$, where K can be any positive constant. Letting $K \to 0$, we get a contradiction.

Hence we have shown $x(t) \leq 0$ for all t. However, this clearly contradicts the terminal condition $x(2) \geq e^4 + \frac{1}{4}$. Accordingly, the problem has no solution.

THE END