16th Hilbert problem: computation of Lyapunov values and limit cycles in two-dimensional dynamical systems

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http://www.math.spbu.ru/user/nk/

http://www.math.spbu.ru/user/nk/Limit_cycles_Focus_values.htm

tutorial last version: http://www.math.spbu.ru/user/nk/PDF/Limit_cycles_Focus_values.pdf

History: existence, number & computation of limit cycles

1900: 16th Hilbert problem (second part)
Number and mutual disposition of limit cycles for

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 y + \ldots$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y + \ldots$$

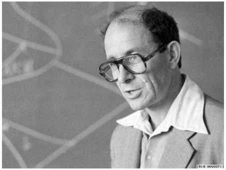
Problem is not solved even for quadratic systems (QS):

- **N.N. Bautin** 1949-1952: 3 limit cycles (LCs) [around one focus]
- **I.G. Petrovskii, E.M. Landis** 1955–1959: only 3 LCs
- **L. Chen & M. Wang, S. Shi** 1979-80: 4 LCs [(1,3), 2 focuses]
- **R. Bamon** 1985: number of LCs in QS is finite
- **P. Zhang** 2001: two focuses ⇒ only (1,n) distribution

Number of limit cycles $H(n)$: $H(2) \geqslant 4$, $H(3) \geqslant 13$
$H(n)$ grows at least as $\frac{(n+2)^2 \ln(n+2)}{2\ln 2}$ for large n (Han&Li,2012).

Computation (visualization) of limit cycles

Small-amplitude limit cycles: only analytical methods
(Lyapunov values: weak focus & Andronov-Hopf bifurcation)
Normal-amplitude limit cycles: analytical&numerical methods



V. Arnold (2005):To estimate the number of LCs of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.&Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!... The fact that this did not occur suggests that the above-mentioned domains are, apparently, small.

Numerical methods: nested cycles are hidden oscillations

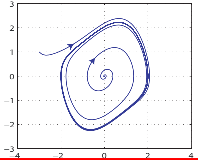
Survey: Leonov G.A., Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, International Journal of Bifurcation and Chaos, 23(1), 2013, art. no. 1330002

Computation of oscillations and attractors

self-excited attractor localization: *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it.

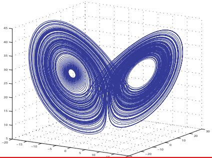
Van der Pol

$$\dot{x} = y$$
$$\dot{y} = -x + \varepsilon(1-x^2)y$$



Lorenz

$$x = -\sigma(x - y)$$
$$y = rx - y - xz$$
$$z = -bz + xy$$

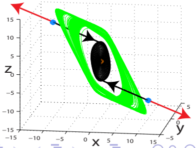


hidden attractor: *if basin of attraction does not intersect with a small neighborhood of equilibria* [Leonov, Kuznetsov, Vagaitsev, *Phys. Lett. A*, 2011]

✓ *standard computational procedure* does not work: all equilibria are stable or not in the basin of attraction

✓ integration with random initial data does not work: basin of attraction is small, system's dimension is large

How to choose initial data in the attraction domain?

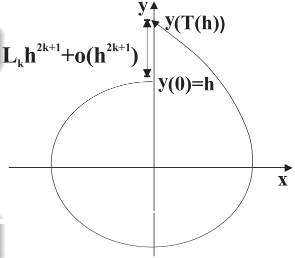


Lyapunov value (focus value, Poincare-Lyapunov constant or quantity)

$$\dot{x} = f_{10}x + f_{01}y + f(x,y) \qquad \text{eig}(A) = \text{eig}\begin{pmatrix} f_{10} & f_{01} \\ g_{10} & g_{01} \end{pmatrix} = \pm i\omega_0 : \quad \widetilde{L}_1 = 0$$
$$\dot{y} = g_{10}x + g_{01}y + g(x,y)$$

$$f = \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big), \quad g = \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big)$$

Poincare map: $\mathrm{L}(h) = y(T(h),h) - h$



Solution with suff. small h:

$x(t,h) = x(t,0,h),\ y(t,h) = y(t,0,h)$

$T(h)$ — return time

$$y(T(h),h) = h(1+\widetilde{L}_1) + \widetilde{L}_2 h^2 + \widetilde{L}_3 h^3 + \widetilde{L}_4 h^4 + .. + o(h^n)$$

First nonzero $\widetilde{L}_i$ has odd index: $\mathrm{L}(h)\mathrm{L}(-h) \leqslant 0$

Lyapunov quantity $\mathrm{L}_k \overset{\text{def}}{=} \widetilde{L}_{2k+1}$ (first $\neq 0$) : $y(T(h),h) - h = \mathrm{L}_k h^{2k+1} + o(h^{2k+1})$
trajectory winding or unwinding & equilibrium stability or instability

A.Lyapunov: similar procedure for dynamical system higher dimension

Lyapunov values: in terms of system's coefficients

To compute general expression of kth Lyapunov value it is necessary to consider expansion upto $2k+1$: $\mathrm{L}_k = \mathrm{L}_k(\{g_{k,j}\}_{k+j=2}^{2k+1}, \{f_{k,j}\}_{k+j=2}^{2k+1})$

$$\dot{x} = -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + \ldots, \ \dot{y} = x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + \ldots.$$

- **1949**, **N. Bautin**:

$$\mathrm{L}_1 = \frac{\pi}{4}(g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})$$

- **1959**, **N. Serebryakova**: $\mathrm{L}_2 = \frac{\pi}{72}($ ...1 page... $)$
- **1968**, **S. Shuko**: first computer program for Lq calculation
- **2008**, **N. Kuznetsov, G. Leonov**: $\mathrm{L}_3 = \frac{\pi}{1728}($ ...4 pages... $)$
- **2010**, **O. Kuznetsova**: $\mathrm{L}_4 = \frac{\pi}{259200}($ ... 45 pages... $)$

To simplify LV expressions, it is often used change of coordinates (complex, polar) & reduction to normal forms (but such reductions is not unique and often laborious).

Survey: Leonov G.A., Kuznetsov N.V., Kudryashova E.V., Cycles of two-dimensional systems: computer calculations, proofs, and experiments, Vestnik St. Petersburg University. Mathematics, 41(3), 2008, 216-250 (doi:10.3103/S1063454108030047)

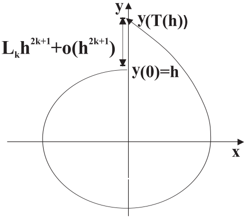
Direct method for Lyapunov values computation

In the study of applied models it is more convenient to perform analysis in "physical" space: in Euclidian coordinates and time domain

$$\dot{x} = -y + f(x,y) = -y + \sum_{k+j=2}^{n} f_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \quad x(t,h) = x(t,0,h)$$
$$\dot{y} = +x + g(x,y) = +x + \sum_{k+j=2}^{n} g_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \quad y(t,h) = y(t,0,h)$$

1. Approximation of solution $x(t,h), y(t,h)$

$$x(t,h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) h^k + o(h^n), \quad y(t,h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) h^k + o(h^n)$$



2. Approximation of return time $T(h):\ x(T(h),h) = 0$

$$T(h) = 2\pi + \Delta T(h) = 2\pi + \sum_{j=1}^{n} \widetilde{T}_j h^j + o(h^n)$$

3. Computation of Lyapunov values $\mathrm{L_k}$: $\{\widetilde{\mathrm{L}}_\mathrm{i}\}_{\mathrm{i}=2}^{2\mathrm{k}} = 0$

$$y(T(h),h) = h + \widetilde{L}_2 h^2 + \widetilde{L}_3 h^3 + \widetilde{L}_4 h^4 + \ldots + o(h^n) \qquad = \mathbf{h} + \mathrm{L}_\mathbf{k} \mathbf{h}^{\mathbf{2k+1}} + \mathbf{o}(\mathbf{h}^{\mathbf{2k+1}})$$

* N.V. Kuznetsov, G.A. Leonov, Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems, Journal of Vibroengineering, 10(4), 2008, 460-467

* G.A. Leonov, N.V. Kuznetsov, E.V. Kudryashova, A direct method for calculating Lyapunov quantities of two-dimensional dynamical systems, Proceedings of the Steklov Institute of Mathematics, 272(Suppl. 1), 2011, 119-127 (doi:10.1134/S008154381102009X)

Direct method for computation of Lq: solution approximation

$$\dot{x} = -y + f(x,y) = -y + \sum_{k+j=2}^{n} f_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \; x(t,h) = x(t,0,h)$$
$$\dot{y} = +x + g(x,y) = +x + \sum_{k+j=2}^{n} g_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \; y(t,h) = y(t,0,h)$$

$$x(t,h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) h^k + o(h^n), \quad y(t,h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) h^k + o(h^n)$$

$$x(t,h) = h\frac{\partial x(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_x(t,h)} = \widetilde{x}_{h^1}(t)h + o(h), \; 0 \le \theta_x(t,h) \le 1$$
$$y(t,h) = h\frac{\partial y(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_y(t,h)} = \widetilde{y}_{h^1}(t)h + o(h), \; 0 \le \theta_y(t,h) \le 1$$

$$k = 1: \quad \dot{\widetilde{x}}_{h^1}(t) = -\widetilde{y}_{h^1}(t), \dot{\widetilde{y}}_{h^1}(t) = \widetilde{x}_{h^1}(t) \;\; f, g\big(x(t,h), y(t,h)\big) = o(h^1)$$

$$x_{h^1}(t,h) = \widetilde{x}_{h^1}(t)h = -h\sin(t), \; y_{h^1}(t,h) = \widetilde{y}_{h^1}(t)h = h\cos(t)$$

Let $x_{h^{k-1}}(t,h) = \sum_{i=1}^{k-1} \widetilde{x}_{h^k}(t)h^k, \, y_{h^{k-1}}(t,h) = \sum_{i=1}^{k-1} \widetilde{y}_{h^k}(t)h^k$ known f-ns(t,h)

$$x_{h^k}(t,h) = x_{h^{k-1}}(t,h) + \widetilde{x}_{h^k}(t)h^k + o(h^k), \; y_{h^k}(t,h) = y_{h^{k-1}}(t,h) + \widetilde{y}_{h^k}(t)h^k + o(h^k)$$
$$f\big(x_{h^{k-1}}(t,h) + o(h^{k-1}), y_{h^{k-1}}(t,h) + o(h^{k-1})\big) = O(h^{k-1}) + u^f_{h^k}(t)h^k + o(h^k)$$
$$g\big(x_{h^{k-1}}(t,h) + o(h^{k-1}), y_{h^{k-1}}(t,h) + o(h^{k-1})\big) = O(h^{k-1}) + u^g_{h^k}(t)h^k + o(h^k)$$

$\Rightarrow u^{f,g}_{h^k}(t) = u^{f,g}_{h^k}\big(\{\widetilde{x}_{h^m}(t), \widetilde{y}_{h^m}(t)\}_{m \le k-1}\big)$ known functions (t)

$$k-1 \to k: \quad \dot{\widetilde{x}}_{h^k}(t) = -\widetilde{y}_{h^k}(t) + u^f_{h^k}(t), \; \dot{\widetilde{y}}_{h^k}(t) = \widetilde{x}_{h^k}(t) + u^g_{h^k}(t)$$

Direct method for computation of Lq: time constants & Lq

$x_{h^n}(t,h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) h^k : \quad x_{h^n}\left(t, h, \{f_{ij}, g_{ij}\}_{i+j=2}^{n}\right) :$ known f-n(t)

$y_{h^n}(t,h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) h^k : \quad y_{h^n}\left(t, h, \{f_{ij}, g_{ij}\}_{i+j=2}^{n}\right) :$ known f-n(t)

$$T(h) = 2\pi + \Delta T(h) = 2\pi + \sum_{j=1}^{n} \widetilde{T}_j h^j + o(h^n) : \quad x(2\pi + \Delta T(h), h) = 0$$

$$\widetilde{x}_{h^k}(2\pi + \Delta T(h)) = \widetilde{x}_{h^k}(2\pi) + \sum_{m=1}^{n} \widetilde{x}_{h^k}^{(m)}(2\pi) \frac{(\Delta T(h))^m}{m!} + o\big((\Delta T(h))^n\big)$$

$$\widetilde{y}_{h^k}(2\pi + \Delta T(h)) = \widetilde{y}_{h^k}(2\pi) + \sum_{m=1}^{n} \widetilde{y}_{h^k}^{(m)}(2\pi) \frac{(\Delta T(h))^m}{m!} + o\big((\Delta T(h))^n\big)$$

$$x(2\pi + \Delta T(h), h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(2\pi + \Delta T(h)) h^k / k! + o(h^n) = 0$$

$$y(2\pi + \Delta T(h), h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(2\pi + \Delta T(h)) h^k / k! + o(h^n) = \sum_{k=1}^{n} \widetilde{L}_k + o(h^n)$$

h :	$0 = \widetilde{x}_{h^1}(2\pi)$	$\widetilde{L}_1 = \widetilde{y}_{h^1}(2\pi)$
h^2 :	$0 = \widetilde{x}_{h^2}(2\pi) + \widetilde{x}'_{h^1}(2\pi)\widetilde{T}_1$	$\widetilde{L}_2 = \widetilde{y}_{h^2}(2\pi) + \widetilde{y}'_{h^1}(2\pi)\widetilde{T}_1$
...	...	...
h^n :	$0 = \widetilde{x}_{h^n}(2\pi) + .. + \widetilde{x}'_{h^1}(2\pi)\widetilde{T}_{n-1}$	$\widetilde{L}_n = \widetilde{y}_{h^n}(2\pi) + .. + \widetilde{y}'_{h^1}(2\pi)\widetilde{T}_{n-1}$

$\widetilde{T}_{k-1} = \widetilde{T}_{k-1}(\{f_{ij}, g_{ij}\}_{i+j=2}^{k})$

$\widetilde{L}_k = L_k(\{T_i\}_{i=1}^{k-1}, \{f_{ij}, g_{ij}\}_{i+j=2}^{k})$

Lyapunov quantity $\mathrm{L}_k \overset{\text{def}}{=} \widetilde{L}_{2k+1}$ (first $\neq 0$) : $y(T(h), h) - h = \mathrm{L}_k h^{2k+1} + o(h^{2k+1})$

Symbolic computation: solution appr-n, time constants & Lq

```
function [L,T,xt,yt] = fLQ_kuzleo(fxy,gxy,N)
syms x y h t 'real'
  NL=2*N+1; Nfg=NL; % CREATE SYMBOLIC REPRESENTATION
  xt_s(1:Nfg-1)=0*h; yt_s(1:Nfg-1)=0*h; xth_s=0*t; yth_s=0*t;
  for n=1:Nfg %1. Create solution as a series of h (x(0,h)=0; y(0,h)=h)
     xt_s(n)=sym(['xt_',int2str(n)],'real'); xth_s=xth_s+xt_s(n)*h^n;
     yt_s(n)=sym(['yt_',int2str(n)],'real'); yth_s=yth_s+yt_s(n)*h^n;
  end
  disp(['NL=',int2str(NL)]); % To calculate L_m , set NL= 2m+1
  sT_h_cur=0; %2. Create crossing time T (x(T,h)=0,y(T,h)>0 ) as a series of h
  for i=1:NL-1
    sT_h(i,1)=sym(['T',int2str(i)],'real');
    sT_h_cur=sT_h_cur + sT_h(i,1)*h^i;
  end;
  % CALCULATION OF LYAPUNOV QUANTITIES
  %1. Calculation x(t,h) y(t,h) as series in terms of  t
  ugt(1:Nfg)=0*t; xt(1:Nfg)=0*t; yt(1:Nfg)=0*t;
  %  solution of the first approximation system
  xt(1)=-sin(t); yt(1)=cos(t); xt_cur=xt(1)*h; yt_cur=yt(1)*h;
  for i=2:NL
    %create approx-n of right-hand sides of the system depending on t
    uft_s=subs(diff(subs(fxy, [x y], [xth_s yth_s]),h,i)/factorial(i),h,0);
    uft(i)= subs(uft_s, [xt_s yt_s], [xt yt]);
    ugt_s=subs(diff(subs(gxy, [x y], [xth_s yth_s]),h,i)/factorial(i),h,0);
    ugt(i)= subs(ugt_s, [xt_s yt_s], [xt yt]);
    uIt=diff(ugt(i),t)+uft(i); %create approximation of solution depending on t
    Iucos=int(cos(t)*uIt,t); Iucos_t0=(Iucos - subs(Iucos,t,0));
    Iusin=int(sin(t)*uIt,t); Iusin_t0=(Iusin - subs(Iusin,t,0));
    ug0=subs(ugt(i),t,0);
    xt(i)=simplify(cos(t)*ug0+Iucos_t0*cos(t)+Iusin_t0*sin(t)-ugt(i));
    yt(i)=simplify(sin(t)*ug0+Iucos_t0*sin(t)-Iusin_t0*cos(t));
    xt_cur=xt_cur+xt(i)*h^i; yt_cur=yt_cur+yt(i)*h^i;
  end;
```

Symbolic computation: solution appr-n, time constants & Lq

```
%2. Calculation coeffisients of x(t,h) in terms of T_k
xh_cur=subs(xt_cur,t,2*pi);
for k=1:NL
    xh_cur=xh_cur + subs(diff(xt_cur,k,t),t,2*pi)*sT_h_cur^k/factorial(k);
end;
for k=1:NL
    xh(k,1)=subs(diff(xh_cur,k,h)/factorial(k),h,0);
end;
%3.  Find T_k from x_k=0
xh_temp=xh; T_cur=0; T(1,1)=0*x;
for k=2:NL
    T(k-1,1)=solve(xh_temp(k,1),sT_h(k-1,1));
    T_cur=T_cur + T(k-1,1)*h^(k-1);
    xh_temp=subs(xh_temp,sT_h(k-1,1),T(k-1,1));
end;
%4.
yh_cur=subs(yt_cur,t,2*pi);
for k=1:NL
    yh_cur=yh_cur + subs(diff(yt_cur,k,t),t,2*pi)*T_cur^k /factorial(k);
end;
for k=1:NL
    yh(k,1)=subs(diff(yh_cur,k,h)/factorial(k),h,0);
end;
for k=1:N
  L(k)=factor(yh(2*k+1))
end;
```

Example: non isochronous center in Duffing $\dot{x}=-y, \dot{y}=x+x^3$

For solution $(x(t),y(t))$ with i.d. $x_0=0, y_0=h$: $y(t)^2+x(t)^2+\frac{1}{2}x(t)^4=h^2$
$\Rightarrow$ all trajectories are closed and periodic: $y(0)=h, x(0)=x(T(h))=0$

$\Rightarrow$ for $(x<0<y)$: $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{x(1+x^2)} = \frac{1}{-\sqrt{-1+\sqrt{1+2h^2-2y^2}}\sqrt{1+2h^2-2y^2}}$

$$T(h) = 4\int_h^0 \frac{\mathrm{d}y}{-\sqrt{-1+\sqrt{1+2h^2-2y^2}}\sqrt{1+2h^2-2y^2}}$$

$y=h\cos(z) \Rightarrow z=\arccos\frac{y}{h}$, $\mathrm{d}y=-h\sin(z)\mathrm{d}z$

$$T(h)=4\int_0^{\pi/2}\frac{h\sin(z)\mathrm{d}z}{\sqrt{-1+\sqrt{1+2h^2\sin^2 z}}\sqrt{1+2h^2\sin^2 z}}=2\pi+\sum_{k=1}^{n}\widetilde{T}_k h$$

$$\widetilde{T}_1=0,\ \widetilde{T}_2=-\frac{3\pi}{4},\ \widetilde{T}_3=0,\ \widetilde{T}_4=\frac{105\pi}{128},\ \widetilde{T}_5=0,\ \widetilde{T}_6=\frac{1155\pi}{1024},\ldots$$

$$x(t,h)=\widetilde{x}_{h^1}(t)h+\widetilde{x}_{h^2}(t)h^2+\widetilde{x}_{h^3}(t)h^3+\widetilde{x}_{h^4}(t)h^4+\ldots$$

$$y(t,h)=\widetilde{y}_{h^1}(t)h+\widetilde{y}_{h^2}(t)h^2+\widetilde{y}_{h^3}(t)h^3+\widetilde{y}_{h^4}(t)h^4+\ldots$$

$$\widetilde{x}_{h^1}(t)=-\sin(t),\quad \widetilde{y}_{h^1}(t)=\cos(t);\quad \widetilde{x}_{h^2}(t)=\widetilde{y}_{h^2}(t)=0$$

$$\widetilde{x}_{h^3}(t)=\frac{1}{8}\cos(t)^2\sin(t)-\frac{3}{8}t\cos(t)+\frac{1}{4}\sin(t),\quad \widetilde{x}_{h^4}(t)=0.$$

$$\widetilde{y}_{h^3}(t)=-\frac{3}{8}t\sin(t)+\frac{3}{8}\cos(t)-\frac{3}{8}\cos(t)^3;\quad \widetilde{y}_{h^4}(t)=0.$$

$T(h)\neq\mathrm{const}$, $\mathsf{L}_1=\mathsf{L}_2=\ldots=0$: non isochronous center

Classical Poincare-Lyapunov method: Lyapunov function

$$\dot{x} = -y + f(x,y) \qquad f(x,y) = \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n$$
$$\dot{y} = +x + g(x,y) \qquad g(x,y) = \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n$$

$$V(x,y) = \frac{x^2+y^2}{2} + V_3(x,y) + .. + V_{n+1}(x,y) \qquad V_k(x,y) = \sum_{i+j=k} V_{i,j}x^i y^j$$

$$\dot{V}(x,y) = \frac{\partial V(x,y)}{\partial x}(-y+f_n(x,y)) + \frac{\partial V(x,y)}{\partial y}(x+g_n(x,y)) + o\big((|x|+|y|)^{n+1}\big)$$

$$\dot{V}(x,y) = W_3(x,y) + ... + W_{n+1}(x,y) + o\big((|x|+|y|)^{n+1}\big)$$

$$W_k(x,y) = \left(x\frac{\partial V_k(x,y)}{\partial y} - y\frac{\partial V_k(x,y)}{\partial x}\right) + u_k(x,y,\{V_{ij},f_{ij},g_{ij}\}_{i+j<k})$$

It is possible to determine $\{V_{ij}\}_{i+j=k}$ for k=3,.. step by step so that $\dot{V}(x,y) = w_1(x^2+y^2)^2 + w_2(x^2+y^2)^3 + ..,$ while $w_{1,..,k-1}=0$: solve a system of $(k+1)$ linear equations. Uniqueness if $V_{(m+1)\,(m+1)} = 0,$ for m odd, $V_{(m)\,(m+2)} + V_{(m+2)\,(m)} = 0,$ for m even

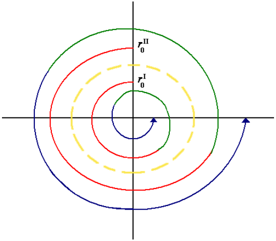
Poncare-Lyapunov constant $\stackrel{\text{def}}{=}$ first $w_m \neq 0 \quad (2\pi w_m = \mathrm{L})$

Lyapunov values & small limit cycles:

Andronov-Hopf bifurcation, cyclicity and center problems

$\dot{x} = f_{10}x + f_{01}y + f(x,y), \quad \dot{y} = g_{10}x + g_{01}y + g(x,y)$

Solution $x(t,h) = x(t,0,h)$, $y(t,h) = y(t,0,h)$, return time $T(h)$



Small limit cycles from weak focus:

$\mathrm{L}_0 = \widetilde{L}_1 = 0, \mathrm{L}_1 = \widetilde{L}_3 > 0$

$y(T(h),h) - h = \mathrm{L}_1 h^3 + o(h^3)$

$g_{01}^{\varepsilon} = g_{01} + \varepsilon_1, \ g_{03}^{\varepsilon} = g_{03} + \varepsilon_3$

$\mathrm{L}_0^{\varepsilon} = \widetilde{L}_1^{\varepsilon} < 0 < \mathrm{L}_1^{\varepsilon} = \widetilde{L}_3^{\varepsilon}, \ |\mathrm{L}_0^{\varepsilon}| << |\mathrm{L}_1^{\varepsilon}|$

$y(T(h),h) - h = \widetilde{L}_1^{\varepsilon} h + \widetilde{L}_2^{\varepsilon} h^2 + \widetilde{L}_3^{\varepsilon} h^3 + o(h^3):$

$\exists h_1, h_2 : y(T(h_1),h_1) - h_1 < 0 < y(T(h_2),h_2) - h_2$

Number of "independent" zeros of Lyapunov values expressions?

Polynomial analysis algebraic methods
Bautin ideal, Groebner basis ...

- C(2)=3, Bautin 1949
- C(3)$\geq$ 11, Zoladek 1995
- C(n)=?,

e.g., a lower bound - Lynch 2005

Large limit cycles of quadratic system: Lienard approach

A. Lienard

$\ddot{x} + f(x)\dot{x} + g(x) = 0$

The classical Lienard theorem:

Let $f(x)$ *be even,* $g(x)$ *be odd,* $xg(x) > 0$ $\forall x \neq 0,\ f(0) < 0,\ f \in C^1(R^1),\ g \in C^1(R^1),$ $f'(x) > 0,\ \forall x > 0,\ \ f(x) \to +\infty$ *as* $x \to +\infty.$

Thm permits to find a unique orbital stable periodic solution.

$$\dot{x} = x^2 + xy + y$$
$$\dot{y} = ax^2 + bxy + cy^2 + \alpha x + \beta y$$

Positively invariant half plane
$\Gamma = \{x > -1,\ r \in R^1\}$

Transformation of Quadratic system to Lienard system

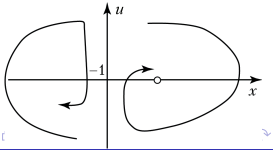
$$\dot{x} = u,\ \ \dot{u} = -f(x)u - g(x) \quad u = \left(y + \frac{x^2}{(x+1)}\right)|x+1|^q$$
$$f(x) = [(2c_2 - b_2 - 1)x^2 - (2 + b_2 + \beta_2) - \beta_2]|x+1|^{q-2},\ \ q = -c_2$$
$$g(x) = [-x(x+1)^2(a_2 x + \alpha_2) + x^2(x+1)(b_2 x - \beta_2) - c_2 x^4]\frac{|x+1|^{2q}}{(x+1)^3}$$

Large limit cycles: asymptotic integration of Lienard eq.

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$
$$f(x) = (Ax^2 + Bx + C)|x+1|^{q-2},$$
$$g(x) = (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}.$$

$$f(x) = \left(A + O\left(\tfrac{1}{|x|}\right)\right)|x|^q,\ \ g(x) = \left(C + O\left(\tfrac{1}{|x|}\right)\right)x|x|^{2q}$$
$$FdF + \frac{\left(A+O\left(\frac{1}{|x|}\right)\right)}{(q+1)}Fd(x^{q+1}) + \frac{\left(C+O\left(\frac{1}{|x|}\right)\right)}{(q+1)}(x)^{q+1}d(x^{q+1}) = 0$$
$$FdF + \frac{A}{(q+1)}Fd(x^{q+1}) + \frac{C}{(q+1)}(x^{q+1})d(x^{q+1}) = 0.$$
$$z = x^{q+1}:\ \ F\frac{dF}{dz} + \frac{A}{(q+1)}F + \frac{C}{(q+1)}z = 0$$
$$\ddot{z} + \frac{A}{(q+1)}\dot{z} + \frac{C}{(q+1)}z = 0$$

Theorem. Boundedness of $x(t), y(t)$ in $\Gamma \Leftrightarrow$ $c_2 \in (0,1), c_2 < b_2 - a_2$ and either $2c_2 > b_2 + 1$ or $2c_2 \le b_2 + 1,\ 4a_2(c_2-1) > (b_2-1)^2$.



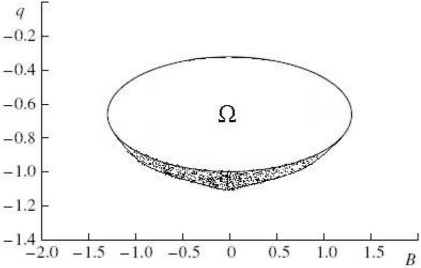
Estimation of parameters domain (Arnold's problem)

$$\dot{x}=a_1x^2+b_1xy+c_1y^2+\alpha_1x+\beta_1y,\ \ \dot{y}=a_2x^2+b_2xy+c_2y^2+\alpha_2x+\beta_2y$$

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \quad f(x)=\big(Ax^2+Bx+C\big)|x+1|^{q-2},$$

$$g(x)=\big(C_1x^4+C_2x^3+C_3x^2+C_4x+C_5\big)\frac{|x+1|^{2q}}{(x+1)^3}.$$

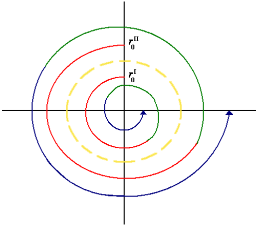
$$A=\tfrac{2}{5}B(q+2),$$
$$C_1=(q+3)\tfrac{B^2}{25}-\tfrac{(1+3q)}{5},$$
$$C_2=\big(15(1-2q)+3B^2\big)\tfrac{1}{25},$$
$$C_3=\tfrac{3(3-q)}{5},\ C_4=1,\ C_5=0.$$
$$\mathrm{L}_3=-\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2+B^2(q-3)]}{20000}$$
$$\Omega:\ \ B^2<-5(q+1)(3q+1),\ B\neq 0$$



One large LC + 3-rd weak focus: 4 LC by small perturbations

G.A. Leonov, N.V. Kuznetsov, Limit Cycles of Quadratic Systems with a Perturbed Weak Focus of Order 3 and a Saddle Equilibrium at Infinity, Doklady Mathematics, 82(2), 2010, 693-696 (doi:10.1134/S1064562410050042)

Four limit cycles in quadratic system



Small limit cycles from weak focus:

$$\mathrm{L}_0 = 0, \quad \mathrm{L}_1 > 0, \quad \widetilde{\mathrm{L}}_0 < 0 < \widetilde{\mathrm{L}}_1, |\widetilde{\mathrm{L}}_0| << |\widetilde{\mathrm{L}}_1|$$

$$y(T(h), h) - h = \mathrm{L}_0 h + \mathrm{L}_1 h^3 + o(h^4)$$

$$L_1 = \frac{-\pi}{4(-\alpha_2)^{5/2}} (\alpha_2(b_2 c_2 - 1) - a_2(b_2 + 2)).$$

$$L_2 = \frac{\pi(b_2-3)(b_2\, c_2-1)^{5/2}}{24(-a_2)^{7/2}(2+b_2)^{7/2}} ((c_2 b_2 + b_2 - 2c_2)(c_2 b_2 - 1) - a_2(c_2 - 1)(1 + 2c_2)^2).$$

$$L_3 = \frac{\pi\sqrt{5}(3c_2 - 1)^{9/2}}{500000(-a_2)^{9/2}} (c_2 - 2)(4c_2^3 a_2 - 3c_2^2 - 3a_2 c_2 - 8c_2 - a_2 + 3).$$

Theorem: Quadratic system has 4 limit cycles, if

$$1/3 < c_2 < 1,\ 1 < b_2 < 3,\ 4a_2(c_2 - 1) > (b_2 - 1)^2,\ b_2 c_2 > 1,$$

$$0 < \beta_2 < \varepsilon,\ \alpha_2 \in (\frac{a_2(2 + b_2)}{b_2 c_2 - 1}, \frac{a_2(2 + b_2)}{b_2 c_2 - 1} + \delta),\ 1 \gg \delta \gg \varepsilon \geq 0.$$

Leonov G.A., Kuznetsova O.A., Lyapunov quantities and limit cycles of two-dimensional dynamical systems. Analytical methods and symbolic computation, Regular and chaotic dynamics, 15(2-3), 2010, 354-377

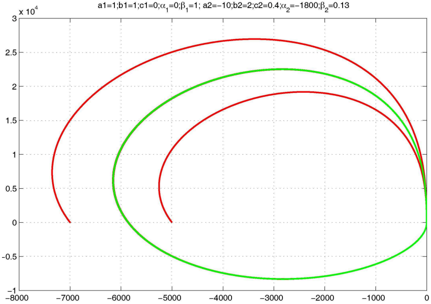
Visualization of 4 normal size limit cycles in QS

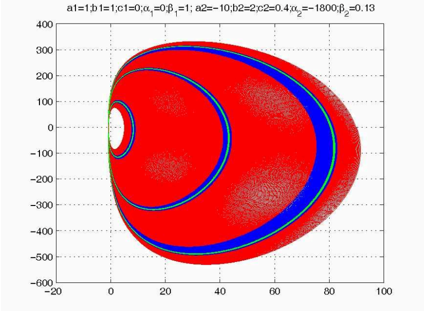
$$\dot{x} = x^2 + xy + y, \quad \dot{y} = ax^2 + bxy + cy^2 + \alpha x + \beta y$$

$$c \in (1/3, 1), \alpha = -\varepsilon^{-1}, bc < 1, b > a + c, 2c < b + 1, 4a(c-1) > (b-1)^2, \beta = 0$$

Theorem. *For sufficiently small ε the system has three limit cycles: one to the left of line $\{x = -1\}$ and two to the right of it.*

(Increase β and get four normal size limit cycles)





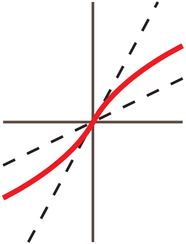
N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Differential equations and Dynamical systems, 21(1-2), 2013, 29-34 (doi:10.1007/s12591-012-0118-6)

Main publications

✓ Leonov G.A., Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, **International Journal of Bifurcation and Chaos**, 23(1), 2013, art. no. 1330002 (doi:10.1007/978-3-642-31353-0_11)
✓ Kuznetsov N.V., Kuznetsova O.A., Leonov G.A., Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, **Differential equations and Dynamical systems**, 21(1-2), **2013**, 29-34 (doi:10.1007/s12591-012-0118-6)
✓ G.A. Leonov, N.V. Kuznetsov, O.A. Kuznetsova, S.M. Seledzhi, V.I.Vagaitsev, Hidden oscillations in dynamical systems, **Transaction on Systems and Control**, 6(2), **2011**, 54–67 (survey)
✓ G.A. Leonov, N.V. Kuznetsov, and E.V. Kudryashova, A Direct Method for Calculating Lyapunov Quantities of Two-Dimensional Dynamical Systems, **Proceedings of the Steklov Institute of Mathematics**, 272(Suppl. 1), **2011**, 119-127 (doi:10.1134/S008154381102009X)
✓ G.A. Leonov, N.V. Kuznetsov, Limit cycles of quadratic systems with a perturbed weak focus of order 3 and a saddle equilibrium at infinity, **Doklady Mathematics**, 82(2), **2010**, 693-696 (doi:10.1134/S1064562410050042)
✓ Kuznetsov N.V., Leonov G.A., Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems, **J. of Vibroengineering**, 10(4), **2008**, 460-467
✓ Leonov G.A., Kuznetsov N.V., Kudryashova E.V., Cycles of Two-Dimensional Systems: Computer Calculations, Proofs, and Experiments, **Vestnik St. Petersburg University. Mathematics**, 41(3), **2008**, 216-250 (doi:10.3103/S1063454108030047)
✓ Leonov G.A., Kuznetsov N.V., Computation of the first Lyapunov quantity for the second-order dynamical system, **IFAC Proceedings Volumes (IFAC-PapersOnline)**, 3(1), **2007**, 87-89 (doi:10.3182/20070829-3-RU-4912.00014)

Hidden oscillations: Aizerman and Kalman conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$, then
is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma < k_2$: $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \ \ k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957: \ \ k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true(Aizerman's: $n \geq 2$, Kalman's: $n \geq 4$)
Periodic solution can exist for nonlinearity from linear stability sector

Aizerman's: I.G.Malkin, N.P.Erugin, N.N.Krasovsky (1952)$n = 2$; V.A.Pliss (1958)$n = 3$

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

Lyapunov exponent: sign inversions, Perron effects, chaos, linearization

$$\begin{cases} \dot{x} = F(x), \; x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?$ $y(t) = 0$ is asympt. stable

! Perron effects: $z(t) = 0$ is exp. stable(unst), $y(t) = 0$ is exp. unstable(st)

Positive largest Lyapunov exponent
doesn't, in general, indicate chaos

Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. Journal of Bifurcation and Chaos, 17(4), 2007, 1079-1107 (doi:10.1142/S0218127407017732)
N.V. Kuznetsov, G.A. Leonov, On stability by the first approximation for discrete systems, 2005 Int. Conf. on Physics and Control, PhysCon 2005, Proc. Vol. 2005, 2005, 596-599