Obligatory exercises MAT 2700

Fall 2005
deadline: 4.11.2005 at 14h00, at the mathematical reception, 7th floor Niels Henrik Abels building

To get passed everything has to be done. No calculator needed

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be a probability space with $K = 3$ elements and let P be a probability on Ω such that $P(\omega_i) > 0$ for all $i = 1, 2, 3$. We will consider the following 3 models for one-period financial markets on Ω :

M1 :

- a bank account with initial value $B_0 = 1$ and interest rate $r = 0$
- $N = 1$:

ω	$S(0,\omega)$	$S(1,\omega)$
ω_1	4	5
ω_2	4	4
ω_3	4	3

M2 :

- a bank account with initial value $B_0 = 1$ and interest rate $r = 0$
- $N = 2$:

ω	$S_1(0,\omega)$	$S_1(1,\omega)$	$S_2(0,\omega)$	$S_2(1,\omega)$
ω_1	4	5	5	2
ω_2	4	4	5	3
ω_3	4	3	5	6

M3 :

- a bank account with initial value $B_0 = 1$ and interest rate $r = \frac{1}{10}$
- $N = 2$:

ω	$S_1(0,\omega)$	$S_1(1,\omega)$	$S_2(0,\omega)$	$S_2(1,\omega)$
ω_1	4	$\frac{11}{2}$	5	$\frac{77}{10}$
ω_2	4	$\frac{22}{5}$	5	$\frac{33}{10}$
ω_3	4	$\frac{33}{10}$	5	$\frac{11}{2}$

Exercise 1 (risk neutral probabilities):

a.) Write down S_1^*, S_2^*, V, V^*, G and G^* for the market M3. What can you say about the discounted values in markets M1 and M2?
b.) What is the definition of a risk neutral probability?
c.) Determine all risk neutral probabilities for the markets M1, M2 and M3.

Exercise 2 (arbitrage):

a.) What is the definition of an arbitrage possibility? Give also the equivalent characterizations of arbitrage in terms of V^* respectively G^*.

b.) Explain from an economical point of view why we don't want arbitrage possibilities in a model.

c.) Give the relationship between arbitrage and risk neutral probabilities. Specify if there are arbitrage possibilities in the markets M1, M2 and M3 and justify your answer. Give a concrete arbitrage possibility for the markets that contain arbitrage (here it is easiest to use the characterization of arbitrage in terms of G^*).

If you computed correctly, you have found that market M2 contains arbitrage. From now on we will exclude market M2 and only consider markets M1 and M3.

Exercise 3 (pricing of contingent claims and complete markets):

a.) What is the definition of a contingent claim and what does it represent?

b.) What do we mean with that a contingent claim is attainable, and how can attainability be characterized in terms of risk neutral probabilities?

c.) What is the arbitrage free price of an attainable claim, and how can this price be expressed with the help of a risk neutral probability?
d.) Let X be a contingent claim. Put

$$V_+(X) := \inf\{E_Q[Y/B_1] : Y \geq X,\ Y \text{ is attainable}\}$$

$$V_-(X) := \sup\{E_Q[Y/B_1] : Y \leq X,\ Y \text{ is attainable}\}.$$

Describe an investment strategy that generates an arbitrage if the price p of X is strictly bigger than $V_+(X)$ resp. strictly smaller than $V_-(X)$. An arbitrage free price p of X consequently has to be such that $p \in [V_-(X), V_+(X)]$.

e.) What is the definition of a complete market? Give the characterization of complete markets in terms of risk neutral probabilities.

f.) Specify all attainable claims in market M1 resp. M3.

g.) Consider the call option

$$X(\omega) = (S_1(1,\omega) - k)^+, \quad k > 0,$$

where $(S_1(1,\omega) - k)^+ := \max\{S_1(1,\omega) - k, 0\}$. Describe what kind of contract you obtain if you buy $X(\omega)$. For $k = 4$, check if $X(\omega)$ is attainable in market M1 and determine $[V_-(X), V_+(X)]$. For $k = 3$ find a replicating portfolio of $X(\omega)$ and give the price of $X(\omega)$ in market M3.

Exercise 4 (utility maximization):

In market M3 consider the following optimal portfolio problem :

$$\max_H \; E[u(V_1)]$$
$$given \qquad V_0 = v,$$

where V_0 is the initial value and V_1 the terminal value of the portfolio corresponding to the strategy H, and the utility function u is given through

$$u(x) = \gamma^{-1} x^{\gamma}, \quad -\infty < \gamma < 1, \, \gamma \neq 0.$$

a.) Go through the first step of the risk neutral computational approach and determine an expression for the value of the optimal portfolio.

b.) Assume now that the probability P in market M3 is given through $P(\omega_1) = 1/6$, $P(\omega_2) = 1/6$, $P(\omega_3) = 2/3$, and put $\gamma = 1/2$. Compute the value of the optimal portfolio and determine the corresponding optimal trading strategy.

Exercise 5 (filtrations and martingales):

a.) Let $S(t,\omega), t = 0, ..., T$ be a stochastic process. Describe how one obtains the 'filtration generated by $S(t,\omega)$'. What does this filtration represent when $S(t,\omega)$ models the risky securities in a multiperiod securities market. Specify the requirement on the trading strategy in order to be consistent with the modeling of the information stream.

b.) For a given filtration $\mathcal{F}_t$, $t = 0, ..., T$, give the definition of a martingale. Show that for a given random variable X the stochastic process defined by

$$Y(t,\omega) := E[X \,|\mathcal{F}_t], \quad t = 0, ..., T,$$

is a martingale.