Solution, Problem 1

a) We have $S_1^*(1,\omega_1) = \dfrac{\frac{10}{3}}{\frac{10}{9}} = 3$, $S_1^*(1,\omega_2) = 1$, $S_1^*(1,\omega_3) = 5$.

Hence $Q = (q_1, q_2, q_3)$ is a risk neutral prob. measure iff

$$q_1 + q_2 + q_3 = 1$$
$$(3-2)q_1 + (1-2)q_2 + (5-2)q_3 = 0$$

The matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & 0 \end{array}\right] \; -1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -1 \end{array}\right] \cdot -\tfrac{1}{2}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & \tfrac{1}{2} \end{array}\right] \; -1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \tfrac{1}{2} \\ 0 & 1 & -1 & \tfrac{1}{2} \end{array}\right],$$

which gives $q_1 = \frac{1}{2} - 2q_3$, $q_2 = \frac{1}{2} + q_3$; $0 < q_3 < \frac{1}{4}$

1b) Since there is at least one r.n.p. measure, there is no arbitrage. Since there are more than one r.n.p.m., the market is not complete.

Solution, Problem 1 (contd.)

c) We consider the matrix

$$A = \begin{bmatrix} B_1(\omega_1) & S_1(1,\omega_1) & S_2(1,\omega_1) \\ B_1(\omega_2) & S_1(1,\omega_2) & S_2(1,\omega_2) \\ B_1(\omega_3) & S_1(1,\omega_3) & S_2(1,\omega_3) \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & \frac{10}{3} & S_2(1,\omega_1) \\ \frac{10}{9} & \frac{10}{9} & S_2(1,\omega_2) \\ \frac{10}{9} & \frac{50}{9} & S_2(1,\omega_3) \end{bmatrix}$$

We want the r.n.p.m. to be unique. By 1a) it has to be on the form

$$Q = \left(\tfrac{1}{2} - 2q_3;\ \tfrac{1}{2} + q_3,\ q_3\right) \quad \text{for some } q_3 \in \left(0, \tfrac{1}{4}\right).$$

Now

$$\frac{9}{10} A = \begin{bmatrix} 1 & 3 & x_1 \\ 1 & 1 & x_2 \\ 1 & 5 & x_3 \end{bmatrix} \quad \text{with} \quad x_i = \frac{9}{10} S_2(1,\omega_i),$$

which reduces to

$$\begin{bmatrix} 1 & 3 & x_1 \\ 0 & 1 & \frac{1}{2}(x_1 - x_2) \\ 0 & 0 & -2x_1 + 2x_2 + x_3 \end{bmatrix}.$$

So we need $-2x_1 + x_2 + x_3 \neq 0$.

In addition we need Q to be a r.n.p.m. for the new market, i.e.

$$(x_1 - S_2(0))\left(\tfrac{1}{2} - 2q_3\right) + (x_2 - S_2(0))\left(\tfrac{1}{2} + q_3\right) + (x_3 - S_2(0))\, q_3 = 0$$

i.e.

$$[2x_1 - x_2 - x_3 + 4S_2(0)]\, q_3 = \tfrac{1}{2} x_1 + \tfrac{1}{2} x_2 - S_2(0)$$

In particular, choosing $x_1 = 5$, $x_2 = x_3 = 1 = S_2(0)$, we get

$$q_3 = \frac{1}{6}$$

Hence

$$Q = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$$

is the unique r.n.p.m. for the extended market. Therefore the extended market is free from arbitrage and it is complete.

Solution, Problem 2

2a) $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2, \omega_3\}\}$

$\mathcal{F}_2$ = the family of all subsets of Ω

2b) Since $X^{-1}(0) = \{\omega_3\} \notin \mathcal{F}_1$, we conclude that X is not measurable with respect to $\mathcal{F}_1$

2c) $$E[X \mid \mathcal{F}_1] = \sum_{A \in \mathcal{P}_{\mathcal{F}_1}} \frac{E[XA]}{P(A)} 1_A$$

Hence, if $\omega \in A_1 = \{\omega_1\}$, we get

$$E[X \mid \mathcal{F}_1](\omega) = \frac{3 \cdot \frac{1}{3}}{\frac{1}{3}} = 3$$

If $\omega \in A_2 = \{\omega_2, \omega_3\}$, we get

$$E[X \mid \mathcal{F}_1](\omega) = \frac{1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{2}.$$

2d) We need to verify that

$$E_Q[\Delta S(t) \mid \mathcal{F}_{t-1}] = 0 \quad \text{for } t = 1, 2$$

$t = 1$: $E_Q[\Delta S(1) \mid \mathcal{F}_0] = (1-3)\frac{1}{2} + (5-3)\frac{1}{3} + (5-3)\frac{1}{6}$
$= -1 + \frac{2}{3} + \frac{2}{6} = 0$

$t = 2, \omega \in A_1$: $E_Q[\Delta S(2) \mid \mathcal{F}_1] = \dfrac{0 \cdot \frac{1}{2}}{\frac{1}{2}} = 0$

$t = 2, \omega \in A_2$: $E_Q[\Delta S(2) \mid \mathcal{F}_1] = \dfrac{(4-5)\frac{1}{3} + (7-5)\frac{1}{6}}{\frac{1}{3} + \frac{1}{6}} = 0$ OK

Solutions, Problem 2 (contd.)

2e) We want to find $H \in \mathbb{H}$ such that

$$V_T^{(H)} = X$$

We have

$$V_T = B_T V_T^* = B_T \{ v + G_T^* \} = B_T \{ v + H_1(1) \Delta S^*(1) + H_1(2) \Delta S^*(2) \}$$

$$= v + H_1(1) \Delta S(1) + H_1(2) \Delta S(2)$$

NOTE: $H_1(1)$ is constant, $H_1(2)$ is $\mathcal{F}_1$ - measurable.

For $\omega = \omega_1$ we get

$$v + H_1(1)(1-3) + H_1(2, \omega_1) \cdot 0 = X(\omega_1) = 3$$

i.e. $\underline{\underline{H_1(1) = -\frac{3}{2}}}$

For $\omega = \omega_2$ we get

$$v + H_1(1)(5-3) + H_1(2, \omega_2)(4-5) = X(\omega_2) = 1$$

i.e. $v + 2(-\frac{3}{2}) - H_1(2, \omega_2) = 1$

i.e. $H_1(2, \omega_2) = v - 4 = H_1(2, \omega_3)$

For $\omega = \omega_3$ we get

$$v + H_1(1)(5-3) + H_1(2, \omega_3)(7-5) = X(\omega_3) = 0$$

i.e. $v + 2(-\frac{3}{2}) + (v-4)2 = 0$

i.e. $3v = 3 + 8 = 11$

$$\underline{\underline{v = \frac{11}{3}}}$$

This gives

$$H_0 + 3H_1(1) = v = \frac{11}{3}$$

i.e.

$$H_0 = \frac{11}{3} - 3(-\frac{3}{2}) = \frac{22+27}{6} = \underline{\underline{\frac{49}{6}}}$$

$$H_1(2, \omega_2) = H_1(2, \omega_3) = v - 4 = \frac{49-24}{6} = \underline{\underline{\frac{25}{6}}}$$

($H_1(2, \omega_1)$ is arbitrary).

Solutions, Problem 2 (contd.)

2f) Define

$U_t(W)$ = the maximum expected utility at time T given that the wealth at time t is W and given $\mathcal{F}_t$.

Dynamic programming principle

$$U_t(W) = \max_{\substack{H \\ \mathcal{F}_t\text{-meas.}}} E\left[U_{t+1}\left(B_{t+1}\left\{\frac{W}{B_t} + H \cdot \Delta S^*(t+1)\right\}\right) \mid \mathcal{F}_t\right]$$

We first put $t = T-1 = 1$. Since $U_2(W) = -\exp(-W)$ we then get

$$U_1(W) = \max_{\substack{H \\ \mathcal{F}_1\text{-meas.}}} E\left[-\exp\left(-\{W + H \cdot \Delta S(2)\}\right) \mid \mathcal{F}_1\right]$$

If $\omega = \omega_1$ this gives

$$U_1(w) = \max_H \frac{\frac{1}{3}\left(-\exp\left(-\{w + H \cdot 0\}\right)\right)}{\frac{1}{3}} = \underline{\underline{-\exp(-w)}}$$

If $\omega = \omega_2$ or ω_3 this gives

$$U_1(w) = \max_H \frac{\frac{1}{3}\left(-\exp\left(-\{W + H(4-5)\}\right) - \exp\left(-\{W + H(7-5)\}\right)\right)}{\frac{1}{3} + \frac{1}{3}}$$

Differentiating w.r.t. H we get

$$-\exp(-\{w - H\}) + \exp(-\{w + 2H\})(-2) = 0$$

or

$$-\exp(H) + 2\exp(-2H) = 0$$

or

$$-\exp(3H) + 2 = 0$$

or

$$H = H_1(2, w_2) = H_1(2, w_3) = \underline{\underline{\frac{1}{3}\ln 2}}$$

Hence

$$U_1(w) = \begin{cases} -\exp(-w) & \text{if } w = w_1 \\ -\frac{1}{2}\left[\exp\left(-\left\{w - \frac{\ln 2}{3}\right\}\right) + \exp\left(-\left\{w + \frac{2\ln 2}{3}\right\}\right)\right] & \text{if } w = w_2, w_3 \end{cases}$$

or

$$U_1(w) = -\exp(-w)\begin{cases} 1 & \text{if } w = w_1 \\ \underbrace{\frac{1}{2}\left[\exp\left(\frac{\ln 2}{3}\right) + \exp\left(\frac{2\ln 2}{3}\right)\right]}_{A} & \text{if } w = w_2, w_3 \end{cases}$$

We now proceed to find $U_0(w)$:

$$U_0(w) = \max_h E\left[U_1(w + h\,\Delta S(1))\right]$$

$$= \max_h \left\{-\exp(-\{w + (1-3)h\}) \cdot \frac{1}{3}\right.$$

$$\left. -\exp(-\{w + (5-3)h\}) \cdot A \cdot \frac{2}{3}\right\}$$

$$= \max_h -\exp(-w)\left\{\frac{1}{3}\exp(2h) + \frac{2}{3}A\exp(-2h)\right\}$$

First order conditions

$$2\exp(2h) - 4A\exp(-2h) = 0$$

$$\exp(4h) = -2A$$

$$\underline{\underline{h_{max} = \frac{1}{4}\ln(-2A) = H_1(1)}}$$

It remains to find $H_0(1)$, $H_0(2,\omega)$:

Since

$$v = V_0 = H_0(1) + H_1(1)\,S(0) = H_0(1) + 3H_1(1),$$

this determines $H_0(1)$.

Hence

$$\text{(i)} \qquad V_1 = H_0(1) + H_1(1)\,S(1,\omega)$$

Since H is self-financing we also have

$$\text{(ii)} \qquad V_1 = H_0(2) + H_1(2)\,S(1,\omega)$$

Combining (i) and (ii) we find $H_0(2)$. $\square$