SOLUTIONS, Exam MAT2700, 12 December 2011

PROBLEM 1

1a)
$$\delta(k) = \begin{cases} 0 & \text{if } k \in \mathbb{K} \\ \infty & \text{if } k \notin \mathbb{K} \end{cases}$$

Hence $\tilde{\mathbb{K}} = \mathbb{K} = [0, \infty) \times [0, \infty)$.

1b) We must verify that

$$E_{Q_k}\left[\frac{R_n + k_n}{1 + \delta(k)}\right] = 0 \quad ; \quad n = 1, 2$$

i.e. that

$$E_{Q_k}[R_n] = -k_n \, ; \quad n = 1, 2$$

i.e.

$$0.2(8 - 40k_1 - 100k_2) - 0.2(11 + 100k_1 - 60k_2) + 0.05(12 - 60k_1 + 160k_2) = -31\,k_1$$

and

$$0.15(8 - 40k_1 - 100k_2) - 0.1(12 - 60k_1 + 160k_2) = -31\,k_2 .$$

1c) In the market $\mathcal{M}_k$ we maximize

$$E[-\exp(-W)] \quad \text{subject to} \quad E_{Q_k}[W] = v.$$

Lagrange multiplier method gives the first order cond.

$$\exp(-W) - \lambda L_k = 0$$

where

$$L_k = \frac{Q_k}{P}.$$

Hence
$$\hat{W}_k = -\ln(\lambda L_k) = -\ln\lambda - \ln L_k$$

The condition $\quad E_{Q_k}[\hat{W}_k] = v$

gives
$$E[L_k(-\ln(\lambda L_k))] = v$$

or
$$E[L_k(-\ln\lambda - \ln L_k)] = v.$$

or
$$-\ln\lambda = E[L_k \ln L_k] + v$$

Therefore
$$\hat{W}_k = E[L_k \ln L_k] - \ln L_k + v.$$

Let $F^{(k)} = (F_1^{(k)}, F_2^{(k)})$ be the corresponding portfolio.

1 d) We now seek $k \in \tilde{\mathbb{K}}$ such that

$$J_k(v) = E[-\exp(-\hat{W}_k)]$$

is minimized. Since

$$J_k(v) = \tfrac{1}{3}\left(-\exp(-\hat{W}_k(\omega_1)) - \exp(-\hat{W}_k(\omega_2)) - \exp(-\hat{W}_k(\omega_3))\right)$$

we get

$$\frac{\partial}{\partial k_1} J_k(v) = \frac{1}{3}\sum_{i=1}^{3} \exp(-\hat{W}_k(\omega_i))\frac{\partial}{\partial k_1}(-\hat{W}_k(\omega_i)) = 0$$

$$\frac{\partial}{\partial k_2} J_k(v) = \frac{1}{3}\sum_{i=1}^{3} \exp(-\hat{W}_k(\omega_i))\frac{\partial}{\partial k_2}(-\hat{W}_k(\omega_i)) = 0$$

for the determination of $\hat{k}_1, \hat{k}_2$.

SOLUTIONS, PROBLEM 2

2a) $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$

$\mathcal{F}_2$ = all subsets of Ω

2b) and 2d) The equations for a martingale measure

$$Q = (q_1, q_2, q_3, q_4)$$

are:

$$q_1 + q_2 + q_3 + q_4 = 1$$

$$E_Q[S_1] = S_0 \Rightarrow 4q_1 + 2q_2 + 4q_3 + 2q_4 = 3$$

$E_Q[S_2 | \mathcal{F}_1] = S_1 \Rightarrow$

$\omega \in A_1 = \{\omega_1, \omega_3\}$: $\dfrac{7q_1 + 3q_3}{q_1 + q_3} = 4$

$\omega \in A_2 = \{\omega_2, \omega_4\}$: $\dfrac{3q_2 + q_4}{q_2 + q_4} = 2$

This gives the system (in matrix form)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 4 & 2 & 3 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \quad -4 \quad -3$$

Elementary row operations reduce this to

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & -3 & -4 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & -3 & -4 & -3 & -3 \end{array}\right] \begin{array}{cc} 2 & 3 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & -4 & -6 & -3 \end{array}\right]$$

which gives $q_4 = \frac{1}{4}$, $q_3 = \frac{3}{8}$, $q_2 = \frac{1}{4}$, $q_1 = \frac{1}{8}$, i.e.

$$\underline{\underline{Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)}}$$

This is the only solution of the system and hence the only martingale measure.

2c) $\omega \in A_1 = \{\omega_1, \omega_3\} \Rightarrow E[Y|\mathcal{F}_1] = \dfrac{36 \cdot \frac{1}{8} + 4 \cdot \frac{3}{8}}{\frac{1}{8} + \frac{3}{8}} = 12$

$\omega \in A_2 = \{\omega_2, \omega_4\} \Rightarrow E[Y|\mathcal{F}_1] = \dfrac{9 \cdot \frac{1}{4} + 9 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = 9$

e) First we seek $H = (H_0(2,\omega_i), H_1(2,\omega_i))$; $i = 1, \dots, 4$ s.t.

$\omega = \omega_1$: $H_0(2,\omega_1) + 7H_1(2,\omega_1) = 36$
$\omega = \omega_2$: $H_0(2,\omega_2) + 3H_1(2,\omega_2) = 9$
$\omega = \omega_3$: $H_0(2,\omega_3) + 3H_1(2,\omega_3) = 4$
$\omega = \omega_4$: $H_0(2,\omega_4) + 1 \cdot H_1(2,\omega_4) = 9$

In addition we require that H is predictable, so that $H_0(2)$ must be $\mathcal{F}_1$ - measurable, i.e.

$$H_0(2,\omega_1) = H_0(2,\omega_3), \quad H_1(2,\omega_1) = H_1(2,\omega_3)$$
$$H_0(2,\omega_2) = H_0(2,\omega_4), \quad H_1(2,\omega_2) = H_1(2,\omega_4)$$

Substituting these into the first 4 equations we get the system

$$H_0(2,\omega_1) + 7H_1(2,\omega_1) = 36$$
$$H_0(2,\omega_2) + 3H_1(2,\omega_2) = 9$$
$$H_0(2,\omega_1) + 3H_1(2,\omega_1) = 4$$
$$H_0(2,\omega_2) + H_1(2,\omega_2) = 9$$

These are 2 systems with 2 unknowns.
We get the solutions

$H_0(2,\omega) = -20$, $H_1(2,\omega) = 8$; $\omega \in A_1 = \{\omega_1, \omega_3\}$

$H_0(2,\omega) = 9$, $H_1(2,\omega) = 0$; $\omega \in A_2 = \{\omega_2, \omega_4\}$

2e) (contd.) It remains to find $H_0(1), H_1(1)$.

<u>Method 1</u>. We can use that (by 2c))

$$V_1^{(H)} = E[Y \mid \mathcal{F}_1] = 12 \cdot \chi_{A_1} + 9\,\chi_{A_2},$$

so that we seek $H_0(1), H_1(1)$ such that

$$\in A_1: \quad H_0(1) + H_1(1)\,4 = 12$$
$$\in A_2: \quad H_0(1) + H_1(1)\,2 = 9$$

which gives

$$\underline{\underline{H_0(1) = 6}}, \quad \underline{\underline{H_1(1) = \tfrac{3}{2}}}$$

<u>Method 2</u>. We can use that H is self-financing, so

$$V_1^{(H)} = \hat{V}_1^{(H)},$$

where

$$V_1^{(H)} = H_0(1) + H_1(1)\,S(1) \quad \text{(value at } t=1 \text{ \underline{before} transact.)}$$

and

$$\hat{V}_1^{(H)} = H_0(2) + H_1(2)\,S(1) \quad \text{(value at } t=1 \text{ \underline{after} transact.)}$$

This gives

$$\omega \in A_1: \quad H_0(1) + 4H_1(1) = -20 + 8 \cdot 4 = 12$$
$$\omega \in A_2: \quad H_0(1) + 2H_1(1) = 9 + 0 \cdot 2 = 9$$

which again gives

$$\underline{\underline{H_0(1) = 6}}, \quad \underline{\underline{H_1(1) = \tfrac{3}{2}}}$$

2f) The condition $V_0^{(H)} = v$ is equivalent to the condition

$$E_Q[W] = v \quad \text{when} \quad W = V_T^{(H)}$$

(and $r = 0$), because then

$$E_Q[W] = E_Q[V_T^{(H)}/B_T] = V_0^{(H)} = v.$$

2g) Using Lagrange multiplier we consider

$$W \to u(W) - \lambda L W$$

where $L = \frac{Q}{P} = \left(\frac{1}{2}, 1, \frac{3}{2}, 1\right)$

The first order conditions for optimal $\hat{W}$ are

$$u'(\hat{W}) = \lambda L$$

i.e.

$$\hat{W}^{-\frac{1}{2}} = \lambda L$$

So

$$\hat{W} = (\lambda L)^{-2} = \lambda^{-2} L^{-2}$$

The constraint $E_Q[\hat{W}] = v$ implies

$$\lambda^{-2} E_Q[L^{-2}] = v \quad \text{i.e.} \quad \lambda^{-2} = \frac{v}{E[\frac{1}{L}]}$$

Therefore we get

$$\hat{W} = \frac{v L^{-2}}{E[\frac{1}{L}]} = \frac{6}{7}\left(4, 1, \frac{4}{9}, 1\right) v = \frac{2v}{21}(36, 9, 4, 9)$$

(When $v = \frac{21}{2}$ this gives $\hat{W} = (36, 9, 4, 9)$.)

2h) In 2e) we have found H such that

$$V_T^{(H)} = (36, 9, 4, 9)$$

Since

$$V_T^{(H)} = H_0(T) B_T + \sum_{n=1}^{N} H_n(T) S_n(T)$$

we see that

$$V_T^{(cH)} = c V_T^{(H)}$$

Therefore, if we want to have $V_T^{(\hat{H})} = \frac{2v}{21} V_T^{(H)} = \frac{2v}{21}(36,9,4,9)$,

we choose

$$\hat{H} = \frac{2v}{21} H.$$