MAT2700: Solution to Exercise 1.10

Assume first that H is an arbitrage possibility. Then $V_1(\omega) \geq 0$ for all ω, and $V_1(\omega) > 0$ for at least one ω, and hence $\sum_{\omega\in\Omega} V_1(\omega) > 0$. Multiplying H by a suitable, positive number, we may assume that $\sum_{\omega\in\Omega} V_1(\omega) = 1$.

Consider the vector

$$y = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \\ V_1^*(\omega_1) \\ V_1^*(\omega_2) \\ \vdots \\ V_1^*(\omega_K) \end{bmatrix}$$

If we let B be the matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 1\dots 1 \\ \Delta S_1^*(\omega_1) & \Delta S_2^*(\omega_1) & \dots & \Delta S_N^*(\omega_1) & -1 & 0\dots 1 \\ \Delta S_1^*(\omega_2) & \Delta S_2^*(\omega_2) & \dots & \Delta S_N^*(\omega_2) & 0 & -1\dots 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_1^*(\omega_K) & \Delta S_2^*(\omega_K) & \dots & \Delta S_N^*(\omega_K) & 0 & 0\dots -1 \end{bmatrix}$$

then

$$By = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

To get the x in the exercise, replace H_1 in y by $\begin{bmatrix} H_1 \\ 0 \end{bmatrix}$ if H_1 is positive, and by $\begin{bmatrix} 0 \\ -H_1 \end{bmatrix}$ if H_1 is negative, and do the same with $H_2, \dots, H_N$ (hence the "H-half" of x has twice as many components as the "H-half" of y). The resulting vector x satisfies the condition in the exercise.

For the converse, assume that the matrix problem $Ax = b$ in the excercise has a nonnegative solution. If we put $H_n = x_{2n-1} - x_{2n}$, it is easy to check that H is an arbitrage opportunity (you also need to define $H_0 = -\sum_{n=1}^N H_n S_n(0)$ such that the value of the portfolio at time 0 is 0).