MAT2700 - MANDATORY ASSIGNMENT, FALL 2009; FASIT

REMINDER: The assignment must be handed in before 15:00 on Friday 30 October, 2009, at the reception of the Department of Mathematics, in the 7th floor of Niels Henrik Abels hus, Blindern. Be careful to give reasons for your answers. To have a passing grade you must have correct answers to at least 50% of the questions and moreover have attempted to solve all of them.

Market model. We consider a (single period) market consisting of a probability space $\Omega = \{\omega_1, \omega_2\}$ with $K = 2$, $N = 1$, probability measure

$$P(\omega) = \begin{cases} 1/2, & \omega = \omega_1, \\ 1/2, & \omega = \omega_2, \end{cases}$$

bank account with $B_0 = 1$, $r = 1/10$, and one risky asset, denoted by $S = S_t = S_t(\omega)$,

$$S_0 = 4, \qquad S_1(\omega) = \begin{cases} 3, & \omega = \omega_1, \\ 5 & \omega = \omega_2. \end{cases}$$

Exercise 1.

1a. Are there any dominant trading strategies ? Argue directly (from the definition of a dominant trading strategy).

1a - answer. $H = (H_0, H_1)^T$ is a dominant trading strategy iff

$$\begin{aligned} V_0 &= H_0 + 4H_1 = 0, \\ V_1(\omega_1) &= \frac{11}{10}H_0 + 3H_1 = X_1, \\ V_1(\omega_2) &= \frac{11}{10}H_0 + 5H_1 = X_2, \end{aligned}$$

for some $X_1 > 0, X_2 > 0$. The two first equations give $H_0 = \frac{40}{14}X_1$ and $H_1 = -\frac{10}{14}X_1$. The third equation gives $-\frac{6}{14}X_1 = X_2$, which has no solution for strictly positive X_1, X_2.

1b. Does the law of one price hold? Again, argue directly from the definition.

Date: November 19, 2009.

1b - answer. Let $X = (X_1, X_2)^T$ be arbitrary, and suppose $V_1 = X$, i.e.,

$$V_1(\omega_1) = \frac{11}{10}H_0 + 3H_1 = X_1,$$
$$V_1(\omega_2) = \frac{11}{10}H_0 + 5H_1 = X_2,$$

which has a unique solution $H_0 = (5X_1 - 3X_2)\frac{5}{11}$ and $H_1 = \frac{X_2 - X_1}{2}$. Hence the law of one price holds.

1c. Determine all risk-neutral probability measures Q (if any).

1c - answer. Determine $Q = (Q_1, Q_2)^T$ by solving

$$\frac{30}{11}Q_1 + \frac{50}{11}Q_2 = 4, \quad Q_1 + Q_2 = 1, \quad Q_1, Q_2 > 0,$$

which amounts to requiring in particular that $E_Q[S_1^*] = S_0 = 4$, where

$$S_1^*(\omega) = \frac{10}{11}S_1(\omega) = \begin{cases} \frac{30}{11}, & \omega = \omega_1, \\ \frac{50}{11} & \omega = \omega_2. \end{cases}$$

The unique solution is $Q_1 = \frac{3}{10}$, $Q_2 = \frac{7}{10}$.

1d. Are there arbitrage opportunities.

1d - answer. In view of c), no.

1e. Is the market complete?

1e - answer. In view of c), yes.

Exercise 2.

2a. Let X be a call option with exercise price $e = \frac{9}{2}$. Determine the arbitrage-free price of X.

2a - answer. Price is

$$\begin{aligned} c := E_Q[X/B_1] &= \frac{10}{11}\left(\frac{3}{10}(S_1(\omega_1) - 4.5)^+ + \frac{7}{10}(S_1(\omega_2) - 4.5)^+\right) \\ &= \frac{10}{11}\left(\frac{3}{10}(3 - 4.5)^+ + \frac{7}{10}(5 - 4.5)^+\right) \\ &= \frac{10}{11}\frac{7}{10}\frac{1}{2} = \frac{7}{22} \approx 0.32. \end{aligned}$$

2b. What is the arbitrage-free price of a put option with exercise price $e = \frac{9}{2}$?

2b - answer. The price can be determined DIRECTLY or via the call-put parity:

$$c - p = S_0 - \frac{e}{1+r},$$

where c is the call price and p is the put price. This yields

$$p = \frac{7}{22} - 4 + \frac{9/2}{11/10} = \frac{7 - 4 * 22 + 45 * 2}{22} = \frac{9}{22} \approx 0.41.$$

2c. Regarding a), what is the replicating trading strategy ?

2c - answer. The call X is attainable iff there exists $H = (H_0, H_1)^T$ such that

$$V_1 = H_0 B_1 + H_1 S_1 = X,$$

i.e.,

$$\begin{aligned} \frac{11}{10} H_0 + 3H_1 &= (3 - 4.5)^+ = 0, \\ \frac{11}{10} H_0 + 5H_1 &= (5 - 4.5)^+ = \frac{1}{2}, \end{aligned}$$

which has a unique solution $H_0 = -\frac{15}{22} \approx -0.68$ and $H_1 = \frac{1}{4}$. Check that

$$V_0 = H_0 B_0 + H_1 S_0 = -\frac{15}{22} + \frac{1}{4} 4 = 1 - \frac{15}{22} = \frac{7}{22},$$

which coincides with the call price calculated in 2a).

2d. Compute the return and mean-return of the risky asset. Moreover, compute the state price density.

2d - answer. We have

$$\begin{aligned} R = \frac{S_1 - S_0}{S_0} &= \begin{cases} (3-4)/5 = -1/4, & \omega = \omega_1, \\ (5-4)/5 = 1/4, & \omega = \omega_2. \end{cases} \\ \overline{R} = E[R] &= \frac{1}{2}(-1/4) + \frac{1}{2}(1/4) = 0. \end{aligned}$$

Moreover, the state price density is given by $L = \frac{Q}{P} = \left(\frac{3}{5}, \frac{7}{5}\right)^T$.

2e. In what follows, suppose $H = \left(-\frac{15}{22}, \frac{1}{4}\right)^T$. Determine the trading strategy H' that generates the state price vector L. Compute the beta of the trading strategy H with respect to the trading strategy H'. Provide a financial interpretation of a negative beta. What about a positive beta ?

2e - answer. To determine H' we have to solve

$$V_1'(\omega_1) = \frac{11}{10}H_0' + 3H_1' = L(\omega_1) = \frac{3}{5},$$
$$V_1'(\omega_2) = \frac{11}{10}H_0' + 5H_1' = L(\omega_2) = \frac{7}{5}.$$

The solution is $H_0' = -\frac{6}{11}$ and $H_1' = \frac{2}{5}$. Indeed, $2H_1 = 4/5$, which gives $H_1 = 2/5$. The second equation then says $\frac{11}{10}H_0' = 7/5 - 5\frac{2}{5} = \frac{7-10}{5} = \frac{-3}{5}$ and thus $H_0' = -\frac{6}{11}$. From this we find that

$$V_0' = H_0' + H_1'S_0 = -\frac{6}{11} + \frac{2}{5}4 = \frac{-6*5+8*11}{55} = \frac{58}{55} \approx 1.06.$$

Since

$$V_1'(\omega_1) = \frac{3}{5}, \qquad V_1'(\omega_2) = \frac{7}{5},$$

it follows that

$$R' = \frac{V_1' - V_0'}{V_0'} = \begin{cases} \left(\frac{3}{5} - \frac{58}{55}\right)/(-\frac{58}{55}) = \frac{-25}{58} \approx -0.43, & \omega = \omega_1, \\ \left(\frac{7}{5} - \frac{58}{55}\right)/(-\frac{8}{55}) = -\frac{19}{58} \approx 0.32, & \omega = \omega_2. \end{cases}$$

Moreover,

$$\overline{R'} = E[R'] = \frac{25}{8}\frac{1}{2} + \left(-\frac{19}{8}\right)\frac{1}{2} = \frac{25}{16} - \frac{19}{16} = \frac{6}{16} = \frac{3}{8} \approx 0.375.$$

Since $V_0 = \frac{7}{22} \approx 0.32$,

$$V_1 = \begin{cases} 0, & \omega = \omega_1, \\ \frac{1}{2}, & \omega = \omega_2, \end{cases}$$

so that

$$R = \frac{V_1 - V_0}{V_0} = \begin{cases} -1, & \omega = \omega_1, \\ = \frac{4}{7} \approx 0.57, & \omega = \omega_2, \end{cases}$$

$$\overline{R} = E[R] = -1\frac{1}{2} + \frac{4}{7}\frac{1}{2} = -\frac{3}{14} \approx -0.21,$$

Consequently,

$$\beta = \frac{\overline{R} - r}{\overline{R'} - r} = \frac{29}{14}.$$

Alternatively, by definition,

$$\beta = \frac{\text{cov}(R, R')}{\text{var}(R')}.$$

We have

$$R' - \overline{R'} = \begin{cases} -\frac{11}{29} \approx -0.38, & \omega = \omega_1, \\ \frac{11}{29} \approx 0.38, & \omega = \omega_2, \end{cases}$$

and

$$\text{var}(R') = E\left[\left(R' - \overline{R'}\right)^2\right] = \left(\frac{11}{29}\right)^2 \frac{1}{2} + \left(\frac{11}{29}\right) \frac{1}{2} = \frac{121}{841} \approx 0.14.$$

Moreover,

$$R - \overline{R} = \begin{cases} -\frac{11}{14} \approx -0.79, & \omega = \omega_1, \\ \frac{11}{14} \approx 0.79, & \omega = \omega_2. \end{cases}$$

This gives

$$\left(R - \overline{R}\right)\left(R' - \overline{R'}\right) = \begin{cases} \frac{121}{406} \approx 0.30, & \omega = \omega_1, \\ \frac{121}{406} =\approx 0.30, & \omega = \omega_2. \end{cases}$$

Consequently,

$$\text{cov}(R, R') = E\left[\left(R - \overline{R}\right)\left(R' - \overline{R'}\right)\right] = \frac{121}{406}\frac{1}{2} + \frac{121}{406}\frac{1}{2} = \frac{121}{406} \approx 0.30.$$

and therefore

$$\beta = \frac{\text{cov}(R, R')}{\text{var}(R')} = \left(\frac{121}{406}\right) / \left(\frac{121}{841}\right) = \frac{29}{14} \approx 2.07.$$

Beta is a systematic risk measurement which quantifies the correlation of a security or portfolio to a benchmark index such as the OSEBX. A positive beta indicates that the portfolio follows the market up or down while a negative beta indicates the opposite, i.e., the portfolio generally moves in the opposite direction of the market.

Exercise 3. We consider a market consisting of a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $K = 3$, $N = 1$, probability measure

$$P(\omega) = \begin{cases} 1/3, & \omega = \omega_1, \\ 1/3, & \omega = \omega_2, \\ 1/3, & \omega = \omega_3, \end{cases}$$

bank account with $B_0 = 1$, $r = 1/10$, and one risky asset, denoted by $S = S_t = S_t(\omega)$,

$$S_0 = 4, \qquad S_1(\omega) = \begin{cases} 3, & \omega = \omega_1, \\ 5 & \omega = \omega_2, \\ 7 & \omega = \omega_3. \end{cases}$$

3a. Determine all risk-neutral probabilities Q. Is the market complete ?

3a - answer. $Q = (Q_1, Q_2, Q_3)^T$ is risk-neutral iff $Q > 0$ and $E_Q[S_1^*] = S_0$ or $E_Q[S_1] = B_1 S_0 = \frac{11}{10}4 = \frac{22}{5}$, i.e.,

$$3Q_1 + 5Q_2 + 7Q_3 = \frac{22}{5},$$
$$Q_1 + Q_2 + Q_3 = 1,$$
$$Q_1, Q_2, Q_3 > 0,$$

Multiplying the second equation by -7 and then adding it to the first equation, yields

$$-4Q_1 - 2Q_2 = \frac{22}{5} - 7 = -\frac{13}{5},$$

and thus $Q_2 = \frac{13}{10} - 2Q_1$, which in turn gives $Q_3 = 1 - Q_1 - Q_2 = 1 - Q_1 - \frac{13}{10} + 2Q_1 = -\frac{3}{10} + Q_1$.

Set $Q_1 = q$. Then $Q_2 = \frac{13}{10} - 2q$, which is strictly positive iff $q < \frac{13}{20} \approx 0.65$, and $Q_3 = -\frac{3}{10} + q$, which is strictly positive iff $q > \frac{3}{10} \approx 0.3$.

Risk-neutral probability measures:

$$Q = \left(q, \frac{13}{10} - 2q, -\frac{3}{10} + q\right)^T, \qquad \frac{3}{10} < q < \frac{13}{20}.$$

So there are infinitely many risk-neutral probability measures, in which case there are no arbitrage opportunities. Thus, market is not complete.

Alternatively, taking instead $Q_3 = q$ as the parameter,

$$Q = \left(\frac{13}{10} + q, \frac{7}{10} - 2q, q\right)^T, \qquad 0 < q < \frac{7}{20}.$$

3b. Describe all attainable claims $X = (X_1, X_2, X_3)^T \in \mathbf{R}^3$.

3b - answer. A claim X is attainable iff there exists $H = (H_0, H_1)^T \in \mathbf{R}^2$ such that $V_1 = H_0 B_1 + H_1 S_1 = X$, i.e.,

$$\frac{11}{10}H_0 + 3H_1 = X_1,$$
$$\frac{11}{10}H_0 + 5H_1 = X_2,$$
$$\frac{11}{10}H_0 + 7H_1 = X_3.$$

From the third equation, $\frac{11}{10}H_0 = -X_3 - 7H_1$. Thus, the second equation gives $-2H_1 = X_2 - X_3$ or $H_1 = \frac{X_3 - X_2}{2}$, while the first equation gives $-4H_1 = X_1 - X_3$

or $H_1 = \frac{X_3 - X_1}{4}$. This gives two expressions for H_1, which must be equal, and hence $\frac{X_3-X_2}{2} = \frac{X_3-X_1}{4}$ or

$$X_1 - 2X_2 + X_3 = 0.$$

An alternative proof goes as follows: The contingent claim X is attainable iff $E_Q[X/B_1]$ does not depend on the choice of Q. Let us check when this is the case. We compute

$$\begin{aligned} E_Q[X/B_1] &= \frac{10}{11}\left(X_1 q + X_2\left(\frac{13}{10} - 2q\right) + X_3\left(\frac{-3}{10} + q\right)\right) \\ &= \frac{10}{11}\left(q\,(X_1 - 2X_2 + X_3) + \frac{13}{10}X_2 - \frac{3}{10}X_3\right), \end{aligned}$$

which is independent of q iff $X_1 - 2X_2 + X_3 = 0$.

Exercise 4. Consider the market model introduced on page 1. Let $H = (H_0, H_1)^T$ be an arbitrary trading strategy and denote by V_t the corresponding total portfolio value at time $t = 0, 1$, referring to V_0 as the initial wealth and V_1 as the terminal wealth. We denote by $u(w)$, $w > 0$, the utility function

$$u(w) = \ln w.$$

Use both the direct approach (1. order conditions at a maximum) and the (two-steps) risk-neutral computational approach / Lagrange multiplier method to solve the problem of maximizing expected utility of terminal wealth, with $V_0 = \nu$ for a given positive number ν:

$$\max_{H \in \mathbf{R}^2} E\left[u\left(V_1\right)\right], \qquad V_0 = \nu,$$

where "solve" means finding the optimal trading strategy H.

4 - answer. First, compute

$$S_1^* = S_1/B_1 = \begin{cases} 30/11, & \omega = \omega_1, \\ 50/11, & \omega = \omega_2, \end{cases}$$

$$\Delta S^* = \begin{cases} 30/11 - 4 = -\frac{14}{11}, & \omega = \omega_1, \\ 50/11 - 4 = \frac{6}{11}, & \omega = \omega_2, \end{cases}$$

and

$$V_1^* = \nu + G^* = \nu + H_1 \Delta S^* = \begin{cases} \nu - \frac{14}{11}H_1, & \omega = \omega_1, \\ \nu + \frac{6}{11}H_1, & \omega = \omega_2. \end{cases}$$

We then obtain

$$E[u(V_1)] = E\left[\frac{11}{10}V_1^*\right] = \frac{1}{2}\ln\left(\frac{11}{10}\left(\nu - \frac{14}{11}H_1\right)\right) + \frac{1}{2}\ln\left(\frac{11}{10}\left(\nu + \frac{6}{11}H_1\right)\right),$$

which gives

$$\frac{\partial}{\partial H_1} E[u(V_1)] = -\frac{7}{11\left(\nu - \frac{14H}{11}\right)} + \frac{3}{11\left(\nu + \frac{6H}{11}\right)} = 0$$

and thus

$$H_1 = -\frac{11}{21}\nu = -0.52\nu.$$

Regarding risk-neutral probability approach,

$$u(w) = \ln w \qquad u'(w) = \frac{1}{w},$$

so

$$\frac{1}{w} = y \Leftrightarrow w = I(y) = \frac{1}{w}.$$

We have

$$\nu = E_Q[I(\lambda L)/B_1)/B_1] = E_Q\left[\frac{B_1}{\lambda L B_1}\right] = \frac{1}{\lambda} E_Q\left[\frac{1}{L}\right] = \frac{1}{\lambda} E_Q\left[\frac{P}{Q}\right] = \frac{1}{\lambda},$$

i..e, $\lambda = \frac{1}{\nu}$.

For the optimal wealth, recalling that $L = \frac{Q}{P} = \left(\frac{3}{5}, \frac{7}{5}\right)^T$,

$$W = I(\lambda L/B_1) = \frac{B_1}{\lambda L} = \frac{11}{10}\nu\frac{1}{L} = \frac{11}{10}\nu \begin{cases} \frac{5}{3}, & \omega = \omega_1, \\ \frac{5}{7}, & \omega = \omega_2 \end{cases} = \begin{cases} \frac{11}{6}\nu, & \omega = \omega_1, \\ \frac{11}{14}\nu, & \omega = \omega_2 \end{cases}$$

To determine the optimal H, we have to solve

$$\frac{11}{10}H_0 + 3H_1 = \frac{11}{6}\nu,$$
$$\frac{11}{10}H_0 + 5H_1 = \frac{11}{14}\nu.$$

from which we deduce

$$H_0 = \frac{65}{21}\nu, \qquad H_1 = -\frac{11}{21}\nu.$$