SOLUTIONS OF MAT 2700 MID-TERM ASSIGNMENT
October 2010

Exercise 1 a)

In general we have

$$R_n = \frac{S_n(1) - S_n(0)}{S_n(0)} \quad ; \; n = 1, 2, \dots, N$$

In our case this gives

$$R_1(\omega_1) = \frac{5-3}{3} = \underline{\underline{\frac{2}{3}}}, \quad R_1(\omega_2) = \frac{2-3}{3} = \underline{\underline{-\frac{1}{3}}}$$

Exercise 1b

The risk neutral probability measures $Q = (Q(\omega_1), Q(\omega_2))$ are characterized by

$$E_Q\left[\frac{R_n - r}{1+r}\right] = 0 \quad \text{for all } n = 1, \dots, N$$

Here this gives the equations

$$\begin{cases} Q(\omega_1) + Q(\omega_2) = 1 \\ \frac{2}{3} Q(\omega_1) - \frac{1}{3} Q(\omega_2) = \frac{1}{20} \end{cases}$$

which has the unique solution

$$\underline{\underline{Q(\omega_1) = \frac{23}{60}}}, \quad \underline{\underline{Q(\omega_2) = \frac{37}{60}}}$$

Exercise 1c

Since there exists a risk neutral probability measure Q, there is no arbitrage.

Exercise 1d) Since there are no arbitrages, there cannot be any dominant strategies.

Exercise 1e) Since there is only one risk neutral probability measure, the market is complete.

Exercise 1f) Since there exists only one risk neutral probability measure, the market satisfy the law of one price.

Exercise 2a In the market $\mathcal{M}_K$ the returns are

$$R_1^K(\omega_1) = \frac{2}{3} + K, \quad R_1^K(\omega_2) = -\frac{1}{3} + K,$$

Therefore the equations for a risk neutral probability measure $Q^{(K)} = (q_1, q_2)$ are

$$\begin{cases} q_1 + q_2 = 1 \\ \left(\frac{2}{3}+K\right) q_1 + \left(-\frac{1}{3}+K\right) q_2 = \frac{1}{20} \end{cases}$$

The solution of this system is

$$q_1 = Q^{(K)}(\omega_1) = \underline{\underline{\frac{23}{60} - K}}, \quad q_2 = Q^{(K)}(\omega_2) = \underline{\underline{\frac{37}{60} + K}},$$

where

$$\underline{\underline{0 \leq K < \frac{23}{60}}}.$$

Exercise 2b)

(i) If $K < \frac{23}{60}$ there exists a risk neutral probability measure and hence there is no arbitrage

(ii) If $K \geq \frac{23}{60}$ then no risk neutral probability measures exist. Hence the market has an arbitrage.

Exercise 2c)

(i) If $K < \frac{23}{60}$ there exists a unique risk neutral probability measure and hence the market is complete.

Exercise 2d)

Maximize $E[\ln W]$ over all $W \in \mathbb{R}^2_+$,

subject to

$$E_{Q^{(k)}}\left[\frac{W}{B_1}\right] = v$$

Using the Lagrange multiplier method we first maximize

$$h(W) := E[\ln W] - \lambda E_{Q^{(k)}}\left[\frac{W}{B_1}\right]$$

without constraints.

Now $h(W) = E\left[\ln W - \lambda \frac{Q^{(k)}}{P} \frac{W}{B_1}\right]$

We can maximize

$$g(x) = \ln x - \lambda \frac{Q^{(k)}(\omega)}{P(\omega)} \cdot \frac{x}{B_1(\omega)} \; ; \; x \in \mathbb{R}_+$$

for each ω.

The first order condition is

$$0 = g'(x) = \frac{1}{x} - \lambda \frac{L^{(k)}(\omega)}{B_1(\omega)} \quad ; \quad L^{(k)}(\omega) = \frac{Q^{(k)}(\omega)}{P(\omega)}$$

which gives

$$x = \hat{W}^{(\lambda)}(\omega) = \frac{B_1(\omega)}{\hat{\lambda} L^{(k)}(\omega)}$$

The constraint

$$E\left[L^{(k)} \frac{W^{(\lambda)}}{B_1}\right] = v$$

gives the value of $\lambda = \hat{\lambda}$:

$$E\left[\frac{L^{(k)}}{B_1}\left(\frac{B_1}{\hat{\lambda} L^{(k)}}\right)\right] = v \quad \Rightarrow \quad \hat{\lambda} = \frac{1}{v}.$$

Hence the optimal $W = \hat{W}^{(\hat{\lambda})}$ is given by

$$W^{(\hat{\lambda})}(\omega) = \frac{B_1(\omega)}{\hat{\lambda} L^{(k)}(\omega)} = \frac{v\, B_1(\omega)}{L^{(k)}(\omega)} .$$

With the given values for P, B_1 and $Q^{(k)}$ we have $L^{(k)} = \left(\frac{23}{30} - 2k, \frac{37}{30} + 2k\right)$ and hence

$$W^{(\hat{\lambda})}(\omega_1) = \frac{21 v}{\frac{46}{3} - 40k} = \underline{\underline{\frac{63 v}{46 - 120 k}}}$$

Similarly,

$$\hat{W}^{(\hat{\lambda})}(\omega_2) = \frac{21 v}{\frac{74}{3} + 40k} = \underline{\underline{\frac{63 v}{74 + 120 k}}}$$

$$0 \le k < \frac{23}{60} .$$

<u>Exercise 2e)</u> We want to find $F^{(k)}$ such that

$$v\left(1 + r + F^{(k)}(R_1^{(k)} - r)\right) = \hat{W}^{(\hat{\lambda})}$$

i.e.

(i) $$v\left(1 + \frac{1}{20} + F^{(k)}\left(\frac{2}{3} + k - \frac{1}{20}\right)\right) = \frac{63 v}{46 - 120 k} \qquad (\omega = \omega_1)$$

and

(ii) $$v\left(1 + \frac{1}{20} + F^{(k)}\left(-\frac{1}{3} + k - \frac{1}{20}\right)\right) = \frac{63 v}{74 + 120 k} \qquad (\omega = \omega_2)$$

$$-\frac{20}{60} - \frac{3}{60}$$

Only one of these two equations is necessary to determine $F^{(k)}$. Equation (i) gives

$$\frac{21}{20} + F^{(k)}\left(\frac{37}{60} + k\right) = \frac{63}{46 - 120k}$$

or

$$F^{(k)} = \frac{63\,(7 + 60k)}{(23 - 60k)(37 + 60k)}$$

The corresponding optimal utility is

$$J_k(v) = E[\ln \hat{W}^{(\hat{\gamma})}]$$

$$= \ln v + \ln 63 - \tfrac{1}{2}\ln(46 - 120k) - \tfrac{1}{2}\ln(74 + 120k)$$

<u>Exercise 3</u> We now want to solve the constrained maximum problem

$$\text{maximize} \quad E[\ln(V_1^{(F)})]$$

subject to

$$F \geq 0 \quad \text{and} \quad V_0 = v.$$

To this end, we introduce the markets $\mathcal{M}_k$ in Exercise 2 and we solve the <u>unconstrained</u> problem

$$\begin{cases} \text{maximize} \quad E[\ln V_1^{(F)}] \\ (\text{given that } V_0 = v) \end{cases}$$

in the market $\mathcal{M}_k$.

This was done in Exercise 2.

We now seek $k = \hat{k}$ such that the corresponding optimal portfolio $F^{(\hat{k})}$ satisfies the conditions

(i) $F^{(\hat{k})} \geq 0$

and

(ii) $F^{(\hat{k})} \cdot \hat{k} = 0$

(Here $\delta(k) = \sup_{F \geq 0} (-F) \cdot k = 0$ for all $k \in \tilde{\mathbb{K}} = [0, \infty)$)

Equation (ii) tells us that either
$F^{(\hat{k})} = 0$ or $\hat{k} = 0$.

The case $F^{(\hat{k})} = 0$ is impossible, because $\hat{k} \geq 0$. Hence we must choose

$$\underline{\underline{\hat{k} = 0}}$$

This means that the constraint $F \geq 0$ is not active and $F^{(0)} = \frac{63.14}{46.37} = \frac{63.7}{23.37} > 0$ is the optimal portfolio for the constrained portfolio problem in Exercise 3. The corresponding optimal utility is

$$J_{\mathbb{K}}(v) = E[\ln(\hat{W})]_{k=0} = \underline{\underline{\ln v + \ln 63 - \tfrac{1}{2}\ln 46 - \tfrac{1}{2}\ln 74}}$$

Alternatively, one could find $\hat{k}$ by minimizing $J_k(v)$ over all $k \in [0, \frac{23}{60})$:

Define
$$h(k) = -\ln(46 - 120k) - \ln(74 + 120k)$$
Then
$$h'(k) = \frac{120}{46 - 120k} - \frac{120}{74 + 120k} = \frac{28 \cdot 120}{(46-120k)(74+120k)}$$

> 0 for all $k \in [0, \frac{23}{60})$.
Hence $h(k)$ is minimal when

$$\underline{\underline{k = \hat{k} = 0}}$$

<u>Exercise 4 a)</u> To find all risk neutral probability measures $Q = (q_1, q_2, q_3)$ we solve the system of equations

$$\begin{aligned} q_1 + q_2 + q_3 &= 1 \quad (Q \text{ prob. meas.}) \\ \tfrac{2}{3} q_1 + (-\tfrac{1}{3}) q_2 + \tfrac{1}{3} \cdot q_3 &= 0 \quad (E_Q[R] = 0) \end{aligned}$$

which has the solution

$$\begin{cases} q_1 = Q(\omega_1) = \frac{1}{3} - \frac{2}{3} q_3 \\ q_2 = Q(\omega_2) = \frac{2}{3} - \frac{1}{3} q_3 \\ q_3 = Q(\omega_3) \in (0, \frac{1}{2}) \end{cases}$$

Exercise 4b) A claim $X = (X_1, X_2, X_3)$ is attainable iff

$$E_Q[X]$$

has the same value for all risk neutral probability measures Q.

With Q as in 4a) we get

$$E_Q[X] = \left(\tfrac{1}{3} - \tfrac{2}{3} q_3\right) X_1 + \left(\tfrac{2}{3} - \tfrac{1}{3} q_3\right) X_2 + q_3 X_3$$

$$= \tfrac{1}{3} X_1 + \tfrac{2}{3} X_2 + q_3\left(-\tfrac{2}{3} X_1 - \tfrac{1}{3} X_2 + X_3\right)$$

This is independent of q_3 iff

$$-\tfrac{2}{3} X_1 - \tfrac{1}{3} X_2 + X_3 = 0$$

or

$$\underline{\underline{2X_1 + X_2 - 3X_3 = 0}}.$$

Exercise 4c) We seek (H_0, H_1) such that

$$H_0(1+r) + H_1 S_1(1,\omega) = X(\omega)$$

i.e.

$$\begin{cases} H_0 + 5H_1 = 2 \\ H_0 + 2H_1 = -1 \\ H_0 + 4H_1 = 1 \end{cases},$$

which has the solution $\underline{\underline{H_0 = -3}}$, $\underline{\underline{H_1 = 1}}$.