Solution; Problem 1

a) $X^{-1}(5) = \{\omega_1, \omega_2, \omega_3\}$, $X^{-1}(0) = \{\omega_4, \omega_5\}$

Hence

$$\mathcal{F}_X = \{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}$$

Similarly,

$$Y^{-1}(1) = \{\omega_1\},\ Y^{-1}(2) = \{\omega_2, \omega_3\},\ Y^{-1}(3) = \{\omega_4, \omega_5\}$$

Therefore

$$\mathcal{F}_Y = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_5\}, \{\omega_2, \omega_3, \omega_4, \omega_5\}\}$$

b) $\mathcal{P}_{\mathcal{F}_X} = \{A_1, A_2\}$, where $A_1 = \{\omega_1, \omega_2, \omega_3\}$, $A_2 = \{\omega_4, \omega_5\}$.

$$\mathcal{P}_{\mathcal{F}_Y} = \{B_1, B_2, B_3\}$$

where

$$B_1 = \{\omega_1\},\ B_2 = \{\omega_2, \omega_3\},\ B_3 = \{\omega_4, \omega_5\}$$

c) Y is not measurable w.r.t. $\mathcal{F}_X$ because

$$Y^{-1}(1) = \{\omega_1\} \notin \mathcal{F}_X.$$

d) On $A_1 = \{\omega_1, \omega_2, \omega_3\}$ we have

$$E[Y \mid \mathcal{F}_X](\omega) = \frac{E[Y \mathbb{1}_{A_1}]}{P(A_1)} = \frac{1\cdot\frac{1}{4} + 2\cdot\frac{1}{6} + 2\cdot\frac{1}{6}}{\frac{1}{4}+\frac{1}{6}+\frac{1}{6}} = \frac{3+4+4}{3+2+2} = \underline{\underline{\frac{11}{7}}}$$

On $A_2 = \{\omega_4, \omega_5\}$ we have

$$E[Y \mid \mathcal{F}_X](\omega) = \frac{E[Y \mathbb{1}_{A_2}]}{P(A_2)} = \frac{3\cdot\frac{1}{6} + 3\cdot\frac{1}{4}}{\frac{1}{6}+\frac{1}{4}} = \frac{6 + \overset{3\cdot 3}{\cancel{12}}}{2+3} = \underline{\underline{\frac{1\cancel{8}5}{5}}} = \underline{\underline{3}}$$

Solution, Problem 2

a) $S^*(1,\omega_1) = 3$, $S^*(1,\omega_2) = 4$, $S^*(1,\omega_3) = 6$

b) Put $Q = (q_1, q_2, q_3)$. Then we get the equations

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ -2q_1 - q_2 + q_3 = 0 \end{cases}$$

which reduce to

$$\begin{cases} q_1 \qquad - 2q_3 = -1 \\ \qquad q_2 + 3q_3 = 2 \end{cases}$$

Hence the general solution is

$$Q = (-1 + 2q_3, 2 - 3q_3, q_3); \quad \tfrac{1}{2} < q_3 < \tfrac{2}{3}.$$

c) Since there is <u>at least one</u> risk neutral measure, the market has no arbitrage.

d) Since there is <u>more than one</u> risk neutral measure, the market is not complete.

e) X is attainable iff

$$E_Q\left[\frac{X}{B_1}\right]$$

has the same value for all $Q \in \mathbb{M}$.

With Q as in b) we get

$$E_Q\left[\frac{X}{B_1}\right] = \frac{9}{10}\left[(-1+2q_3)X_1 + (2-3q_3)X_2 + q_3 X_3\right]$$

$$= \frac{9}{10}\left[q_3(2X_1 - 3X_2 + X_3) - X_1 + 2X_2\right]$$

This is independent of q_3 iff

$$2X_1 - 3X_2 + X_3 = 0$$

f) We seek H_0, H_1 such that

$$\frac{10}{9} H_0 + H_1 \frac{30}{9} = 2$$
$$\frac{10}{9} H_0 + H_1 \frac{40}{9} = 1$$
$$\frac{10}{9} H_0 + H_1 \frac{20}{3} = -1$$

This system has the solution

$$H_0 = 4.5, \quad H_1 = -\frac{9}{10}.$$

g) This follows from the computation in $\overset{e}{d}$)

h) Put $F(W_1, W_2, W_3) = E[u(W)] - \lambda_1(2W_1 - 3W_2 + W_3) - \lambda_2(-W_1 + 2W_2 - \frac{10}{9}v$

Then
$$\frac{\partial F}{\partial W_1} = \frac{1}{3} e^{-W_1} - 2\lambda_1 + \lambda_2 = 0$$

$$\frac{\partial F}{\partial W_2} = \frac{1}{3} e^{-W_2} + 3\lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial F}{\partial W_3} = \frac{1}{3} e^{-W_3} - \lambda_1 = 0$$

Together with the equations

$$2W_1 - 3W_2 + W_3 = 0$$
$$-W_1 + 2W_2 = \frac{10}{9} v$$

this gives 5 equations in the 5 unknowns
$W_1, W_2, W_3, \lambda_1, \lambda_2$.

End of Solutions

This is independent of ξ_0 !!!

$$2\xi_1 - 3\xi_2 + \xi_3 = 0$$

4) We seek H_0, H_1 such that

$$\xi_1: \quad H_0 + H_1 \frac{3Q}{4} = 2$$
$$\xi_2: \quad H_0 + H_1 \frac{3Q}{4} = 1$$
$$\xi_3: \quad H_0 - H_1 \frac{3Q}{4} = -1$$

This system has the solution
$H_1 = \frac{?}{?}$, $H_0 = -\frac{5}{16}$

5) This follows from the computation in 4)

6) Put $F(W_1, W_2, W_3) = E[u(W)] - \lambda_1(2W_1 - 3W_2 + W_3) - \lambda_2(W_1 + W_2 - 2W_3 - \frac{10}{3})$

Then $\frac{\partial F}{\partial W_1} = \frac{1}{4} e^{-W_1} - 2\lambda_1 + \lambda_2 = 0$

$$\frac{\partial F}{\partial W_2} = \frac{1}{2} e^{-W_2} + 3\lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial F}{\partial W_3} = \frac{1}{4} e^{-W_3} - \lambda_1 = 0$$

Together with the equations
$2W_1 - 3W_2 + W_3 = 0$
$W_1 + W_2 = \frac{10}{3}$

This gives 5 equations in the 5 unknowns
$W_1, W_2, W_3, \lambda_1, \lambda_2$

End of Solutions