MAT4500 Topology

Brief review of sets and functions

In basic topology there is a lot of manipulation with sets and functions. The textbook has a good introduction to these issues, but it is much too long to be covered in class. The purpose of this little note is to cut the introduction down to a reasonable size by concentrating on what will be most important for us.

Sets and boolean operations

For us, a set will just be a (finite or infinite) collection of mathematical objects (there are problems with this naive point of view if you push things too far, but we are not likely to run into these problems in this course). We shall write $x \in A$ to say that x is an *element* of the set A, and $x \notin A$ to say that x is not an element of A. Two sets are *equal* if they have exactly the same elements, and we say that A is *subset* of B (and write $A \subset B$) if all elements of A are elements of B, but not necessarily vice versa. Note that there is no requirement that A is *strictly* included in B, and hence it is correct to write $A \subset B$ when $A = B$ (in fact, a standard technique for showing that $A = B$ is first to show that $A \subset B$ and then that $B \subset A$). By $\emptyset$ we shall mean the *empty set*, i.e. the set with no elements (you may feel that a set with no elements is a contradiction in terms, but mathematical life would be much less convenient without the empty set).

To specify a set, we shall often use expressions of the kind

$$A = \{a \,|\, P(a)\}$$

which means the set of all objects satisfying condition P. Often it is more convenient to write

$$A = \{a \in B \,|\, P(a)\}$$

which means the set of all elements in B satisfyng the condition P.

If $A_1, A_2, \ldots, A_n$ are sets, their *union* and *intersection* are given by

$$A_1 \cup A_2 \cup \ldots \cup A_n = \{a \mid a \text{ belongs to } \textit{at least one of} \text{ the sets } A_1, A_2, \ldots, A_n\}$$

and

$$A_1 \cap A_2 \cap \ldots \cap A_n = \{a \mid a \text{ belongs to } \textit{all} \text{ the sets } A_1, A_2, \ldots, A_n\},$$

respectively. Remember that two sets are called *disjoint* if they do not have elements in common, i.e. if $A \cap B = \emptyset$.

It is easy to check that unions and intersections are distributive both ways, i.e.

$$B \cap (A_1 \cup A_2 \cup \ldots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n)$$

and

$$B \cup (A_1 \cap A_2 \cap \ldots \cap A_n) = (B \cup A_1) \cap (B \cup A_2) \cap \ldots \cap (B \cup A_n)$$

There are also other algebraic rules for unions and intersections, but most of them are so obvious that we do not need to state them here (an exception is De Morgan's laws which we shall return to in a moment).

The *set theoretic difference* $A \setminus B$ (also written $A - B$) is defined by

$$A \setminus B = \{a \,|\, a \in A, a \notin B\}$$

In many situations we are only interested in subsets of a given set U (often referred to as the *universe*). The *complement* A^c of a set A with respect to U is defined by

$$A^c = U \setminus A = \{a \in U \,|\, a \notin A\}$$

We can now formulate *De Morgan's laws*:

Proposition 1 (De Morgan's laws) *Assume that $A_1, A_2, \ldots, A_n$ are subsets of a universe U. Then*

$$(A_1 \cup A_2 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap \ldots \cap A_n^c$$

and

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^c = A_1^c \cup A_2^c \cup \ldots \cup A_n^c$$

(These rules are easy to remember if you observe that you can distribute the c outside the parentheses on the individual sets provided you turn all $\cup$'s into $\cap$'s and all $\cap$'s into $\cup$'s).

Proof of De Morgan's laws: We prove the first part and leave the second to the reader. The strategy is as indicated above; we first show that any element of the set on the left must also be an elment of the set on the right, and then vice versa.

Assume that $x \in (A_1 \cup A_2 \cup \ldots \cup A_n)^c$. Then $x \notin A_1 \cup A_2 \cup \ldots \cup A_n$, and hence $x \notin A_i$ for all i. This means that $x \in A_i^c$ for all i, and hence $x \in A_1^c \cap A_2^c \cap \ldots \cap A_n^c$.

Assume next that $x \in A_1^c \cap A_2^c \cap \ldots \cap A_n^c$. This means that $x \in A_i^c$ for all i, in other words: $x \notin A_i$ for all i. Thus $x \notin A_1 \cup A_2 \cup \ldots \cup A_n$ which means that $x \in (A_1 \cup A_2 \cup \ldots \cup A_n)^c$. $\square$

Families

A set of sets is usually called a *family*. An example is the family

$$\mathcal{A} = \{[a, b] \mid a, b \in \mathbb{R}\}$$

of all closed and bounded intervals on the real line. Families may seem abstract, but you have to get used to them as they appear in all parts of higher mathematics. We can extend the notions of union and intersection to families in the following way: If $\mathcal{A}$ is a family of sets, we define

$$\bigcup_{A \in \mathcal{A}} A = \{a \mid a \text{ belongs to at least one set } A \in \mathcal{A}\}$$

and

$$\bigcap_{A \in \mathcal{A}} A = \{a \mid a \text{ belongs to all sets } A \in \mathcal{A}\}$$

The distributive laws and the laws of De Morgan extend to this case in the obvious way — e.g.,

$$(\bigcup_{A \in \mathcal{A}} A)^c = \bigcap_{A \in \mathcal{A}} A^c$$

Families are often given as *indexed sets*. This means we we have one basic set I, and that the family consists of one set A_i for each element in I. We then write the family as

$$\mathcal{A} = \{A_i \mid i \in I\},$$

and use notation such as

$$\bigcup_{i \in I} A_i \qquad \text{and} \qquad \bigcap_{i \in I} A_i$$

for unions and intersections

A rather typical example of an indexed set is $\mathcal{A} = \{B_r \mid r \in [0, \infty)\}$ where $B_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$. This is the family of all circles in the plane with centre at the origin.

Cartesian products

Assume that we have sets $A_1, A_2, \ldots, A_n$. The set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$ is called the *cartesian product* of A_1, A_2, $\ldots$, A_n, and is denoted by $A_1 \times A_2 \times \cdots \times A_n$. If all the sets are the same (i.e. $A_i = A$ for all i), we usually write A^n instead of $A \times A \times \cdots \times A$.

We can extend the notion of cartesian products to indexed families. Assume that $\mathcal{A} = \{A_i \mid i \in I\}$ where I is a (possibly infinite) index set. The cartesian product $\prod_{i \in I} A_i$ consist of all expressions[1] $(a_i)_{i \in I}$ where $a_i \in A_i$ for all $i \in I$. If all A_i are equal (i.e. $A_i = A$ for all $i \in I$), we usually write A^I for $\prod_{i \in I} A$. Note that $\mathbb{R}^{\mathbb{N}}$ is the set of all real sequences.

It may seem trivial that $\prod_{i \in I} A_i$ is nonempty when all the A_i's are nonempty. If the set I is infinite, it turns out, however, that this is far from obvious, and that to prove it, one needs a strong (and somewhat controversial) axiom called the *axiom of choice*. See §9 in the textbook for more information.

[1]More formally, the elements in $\prod_{i \in I} A_i$ are functions $a : I \to \bigcup_{i \in I} A_i$ such that $a(i) \in A_i$ for all $i \in I$.

Functions

Functions can be defined in terms of sets, but for our purposes it suffices to think of a function $f : X \to Y$ from X to Y as a *rule* which to each element $x \in X$ assigns an element $y = f(x)$ in Y. If $f(x) \neq f(y)$ whenever $x \neq y$, we call the function *injective* (or *one-to-one*). If there for each $y \in Y$ is an $x \in X$ such that $f(x) = y$, the function is called *surjective* (or *onto*). A function which is both injective and surjective, is called *bijective* — it establishes a one-to-one correspondence between the elements of X and Y.

If A is subset of X, the set $f(A) \subset Y$ defined by

$$f(A) = \{f(a) \mid a \in A\}$$

is called the *image of A under f*. If B is subset of Y, the set $f^{-1}(B) \subset X$ defined by

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is called the *inverse image of B under f*. In topology, images and inverse images of sets play important parts, and it is useful to know how these operations relate to the boolean operations of union and intersection. Let us begin with the good news.

Proposition 2 *Let $\mathcal{B}$ be a family of subset of Y. Then for all functions $f : X \to Y$ we have*

$$f^{-1}(\bigcup_{B\in\mathcal{B}} B) = \bigcup_{B\in\mathcal{B}} f^{-1}(B) \quad \text{and} \quad f^{-1}(\bigcap_{B\in\mathcal{B}} B) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$$

We say that inverse images commute *with arbitrary unions and intersections.*

Proof: I prove the first part; the second part is proved similarly. Assume first that $x \in f^{-1}(\bigcup_{B\in\mathcal{B}} B)$. This means that $f(x) \in \bigcup_{B\in\mathcal{B}} B$, and consequently there must be at least one $B \in \mathcal{B}$ such that $f(x) \in B$. But then $x \in f^{-1}(B)$, and hence $x \in \bigcup_{B\in\mathcal{B}} f^{-1}(B)$. This proves that $f^{-1}(\bigcup_{B\in\mathcal{B}} B) \subset \bigcup_{B\in\mathcal{B}} f^{-1}(B)$.

To prove the opposite inclusion, assume that $x \in \bigcup_{B\in\mathcal{B}} f^{-1}(B)$. There must be at least one $B \in \mathcal{B}$ such that $x \in f^{-1}(B)$, and hence $f(x) \in B$. This implies that $f(x) \in \bigcup_{B\in\mathcal{B}} B$, and hence $x \in f^{-1}(\bigcup_{B\in\mathcal{B}} B)$. □

For forward images the situation is more complicated:

Proposition 3 *Let $\mathcal{A}$ be a family of subset of X. Then for all functions $f : X \to Y$ we have*

$$f(\bigcup_{A\in\mathcal{A}} A) = \bigcup_{A\in\mathcal{A}} f(A) \quad \text{and} \quad f(\bigcap_{A\in\mathcal{A}} A) \subset \bigcap_{A\in\mathcal{A}} f(A)$$

In general, we do not *have equality in the last case. Hence forward images commute with unions, but not always with intersections.*

Proof: To prove the statement about unions, we first observe that since $A \subset \bigcup_{A\in\mathcal{A}} A$ for all $A \in \mathcal{A}$, we have $f(A) \subset f(\bigcup_{A\in\mathcal{A}})A$ for all such A. Since this inclusion holds for all A, we must also have $\bigcup_{A\in\mathcal{A}} f(A) \subset f(\bigcup_{A\in\mathcal{A}})$. To prove the opposite inclusion, assume that $y \in f(\bigcup_{A\in\mathcal{A}} A)$. This means that there exists an $x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x) = y$. This x has to belong to at least one $A \in \mathcal{A}$, and hence $y \in f(A) \subset \bigcup_{A\in\mathcal{A}} f(A)$.

To prove the inclusion for intersections, just observe that since $\bigcap_{A\in\mathcal{A}} A \subset A$ for all $A \in \mathcal{A}$, we must have $f(\bigcap_{A\in\mathcal{A}} A) \subset f(A)$ for all such A. Since this inclusion holds for all A, it follows that $f(\bigcap_{A\in\mathcal{A}} A) \subset \bigcap_{A\in\mathcal{A}} f(A)$. The example below shows that the opposite inclusion does not always hold. $\Box$

Example 1: Let $X = \{x_1, x_2\}$ and $Y = \{y\}$. Define $f : X \to Y$ by $f(x_1) = f(x_2) = y$, and let $A_1 = \{x_1\}, A_2 = \{x_2\}$. Then $A_1 \cap A_2 = \emptyset$ and consequently $f(A_1 \cap A_2) = \emptyset$. On the other hand $f(A_1) = f(A_2) = \{y\}$, and hence $f(A_1) \cap f(A_2) = \{y\}$. This means that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$. ♣.

The problem in this example stems from the fact that y belongs to both $f(A_1)$ and $f(A_2)$, but only as the image of two *different* elements $x_1 \in A_1$ og $x_2 \in A_2$; there is no *common* element $x \in A_1 \cap A_2$ which is mapped to y. This problem disappears if f is injective:

Corollary 4 *Let $\mathcal{A}$ be a family of subset of X. Then for all* injective *functions $f : X \to Y$ we have*

$$f(\bigcap_{A\in\mathcal{A}} A) = \bigcap_{A\in\mathcal{A}} f(A)$$

Proof: The easiest way to show this is probably to apply Proposition 2 to the inverse function of f, but I choose instead to prove the missing inclusion $f(\bigcap_{A\in\mathcal{A}} A) \supset \bigcap_{A\in\mathcal{A}} f(A)$ directly.

Assume $y \in \bigcap_{A\in\mathcal{A}} f(A)$. For each $A \in \mathcal{A}$ there must be an element $x_A \in A$ such that $f(x_A) = y$. Since f is injective, all these $x_A \in A$ must be the same element x, and hence $x \in A$ for all $A \in \mathcal{A}$. This means that $x \in \bigcap_{A\in\mathcal{A}} A$, and since $y = f(x)$, we have proved that $y \in f(\bigcap_{A\in\mathcal{A}} A)$. $\Box$

Taking complements is another operation that commutes with inverse images, but not (in general) with forward images.

Proposition 5 *Assume that $f : X \to Y$ is a function and that $B \subset Y$. Then $f^{-1}(B^c)) = (f^{-1}(B))^c$. (Here, of course, $B^c = Y \setminus B$ is the complement with respect to the universe Y, while $(f^{-1}(B))^c = X \setminus f^{-1}(B)$ is the complemet with respect to the universe X).*

Proof: An element $x \in X$ belongs to $f^{-1}(B^c)$ iff $f(x) \in B^c$. On the other hand, it belongs to $(f^{-1}(B))^c$ iff $f(x) \notin B$, i.e. iff $f(x) \in B^c$. $\Box$

Finally, let us just observe that being disjoint is also a property that is conserved under inverse images; if $A \cap B = \emptyset$, then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Again the corresponding property for forward images does not hold in general.

Partitions and equivalence relation

Most of you probably know about partitions and equivalence relations from other courses, and this section is just intended as a quick refresher.

If X is a set and $\mathcal{P}$ is a family of subset of X such that

(i) $\bigcup_{P\in\mathcal{P}} P = X$

(ii) any two subsets of $\mathcal{P}$ are disjoint,

we call $\mathcal{P}$ a *partition* of X. If we introduce a relation[2] on X by

$$x \sim y \Longleftrightarrow x \text{ and } y \text{ belong to the same set } P \in \mathcal{P},$$

it is easy to check that $\sim$ has the following three properties:

(i) $x \sim x$ for all $x \in X$,

(ii) If $x \sim y$, then $y \sim x$,

(iii) If $x \sim y$ and $y \sim z$, then $x \sim z$.

We say that $\sim$ is the relation *induced by* the partition $\mathcal{P}$.

Let us now turn the tables around and start with a relation on X satisfying conditions (i)-(iii). Any such relation is called an *equivalence relation.* For each $x \in X$, define the *equivalence class* $[x]$ of x by:

$$[x] = \{y \in X \mid x \sim y\}$$

It is an easy exercise (using properties (i)-(iii)) to show that two equivalence classes $[x]$ and $[y]$ are either equal or disjoint, and hence that

$$\mathcal{P} = \{[x] \mid x \in X\}$$

is a partition of X. Hence there is a one-to-one correspondence between partitions and equivalence relations — all partitions induce an equivalence relation, and all equivalence relations define a partition.

If $\sim$ is an equivalence relation on X, we let $X/\sim$ denote the set of all equivalence classes of $\sim$.

Countability

A set A is called *countable* if it possible to make a list $a_1, a_2, \ldots, a_n, \ldots$ which contains all elements of A. This is the same as saying that A is countable if there exists a *surjective* function $f : \mathbb{N} \to A$. Finite sets are obviously countable (you may list the same elements many times) and so is the set $\mathbb{N}$ of all natural numbers. Observe further that the set $\mathbb{Z}$ of all integers is countable — we just use the list

$$0, 1, -1, 2, -2, 3, -3 \ldots$$

[2]Formally, a (binary) relation on X is just a subset R of $X \times X$; we write $x \sim y$ for $(x, y) \in R$.

It is also easy to see that a subset of a countable set must be countable, and that the image $f(A)$ of a countable set is countable (if $\{a_n\}$ is a listing of A, then $\{f(a_n)\}$ is a listing of $f(A)$).

The next result is perhaps more surprising:

Proposition 6 *If the sets $A_1, A_2, \ldots, A_n$ are countable, so is the cartesian product $A_1 \times A_2 \times \cdots \times A_n$.*

Proof: It suffices to prove that if A and B are countable, so is $A \times B$, as the general statement then follows by induction on the number of sets in the product.

Since A and B are countable, there are lists $\{a_n\}$, $\{b_n\}$ containing all the elements of A and B, respectively. But then

$$\{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), (a_4, b_1), (a_3, b_2), \ldots,\}$$

is a list containing all elements of $A \times B$ (observe how the list is made; first we list the (only) element (a_1, b_1) where the indicies sum to 2, then we list the elements $(a_2, b_1), (a_1, b_2)$ where the indicies sum to 3, then the elements $(a_3, b_1), (a_2, b_2), (a_1, b_3)$ where the indicies sum to 4 etc.) $\square$

The same trick can be used to prove the next result:

Proposition 7 *If the sets $A_1, A_2, \ldots, A_n, \ldots$ are countable, so is their union $\bigcup_{n\in\mathbb{N}} A_n$. Hence a countable union of countable sets is itself countable.*

Proof: Let $A_i = \{a_{i1}, a_{i2}, \ldots, a_{in}, \ldots\}$ be a listing of the i-th set. Then

$$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, \ldots\}$$

is a listing of $\bigcup_{i\in\mathbb{N}} A_i$. $\square$

Proposition 6 can also be used to prove that the rational numbers are countable:

Proposition 8 *The set $\mathbb{Q}$ of all rational numbers is countable.*

Proof: According to proposition 6, the set $\mathbb{Z} \times \mathbb{N}$ is countable and can be listed $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$. But then $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ is a list of all the elements in $\mathbb{Q}$ (due to cancellations, all rational numbers will appear infinitely many times in this list, but that doesn't matter). $\square$

Finally, we have:

Proposition 9 *The set $\mathbb{R}$ of all real numbers is* not *countable.*

Proof: (Cantor's diagonal argument) Assume for contradiction that $\mathbb{R}$ is countable and can be listed $r_1, r_2, r_3, \ldots$. Let us write down the decimal expansions of the numbers on the list:

$$\begin{array}{rcl} r_1 & = & w_1.a_{11}a_{12}a_{13}a_{14}\ldots \\ r_2 & = & w_2.a_{21}a_{22}a_{23}a_{24}\ldots \\ r_3 & = & w_3.a_{31}a_{32}a_{33}a_{34}\ldots \\ r_4 & = & w_4.a_{41}a_{42}a_{43}a_{44}\ldots \\ \vdots & \vdots & \qquad\vdots \end{array}$$

(w_i is the integer part of r_i, and $a_{i1}, a_{i2}, a_{i3}, \ldots$ are the decimals). To get our contradiction, we introduce a new decimal number $c = 0.c_1c_2c_3c_4\ldots$ where the decimals are defined by:

$$c_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1 \\ \\ 2 & \text{if } a_{ii} = 1 \end{cases}$$

This number has to be different from the i-th number r_i on the list as the decimal expansions disagree on the i-th place (as c has only 1 and 2 as decimals, there are no problems with nonuniqueness of decimal expansions). This is a contradiction as we assumed that all real numbers were on the list. $\square$