"Decomposition of a linear functional into positive ones"
For use in MAT4410, autumn 2012

Nadia S. Larsen

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The purpose of this note is to prove the following statement:

Lemma 0.1. *Let (X, Σ, μ) be a measure space, let $1 \leq p < \infty$ and let $L^p(X, \mu)$ be the space of real-valued p-integrable functions on X. If φ is a bounded linear functional on $L^p(X, \mu)$, then $\varphi = \varphi_+ - \varphi_-$, where both φ_+ and φ_- are positive bounded functionals.*

Proof. By linearity we have $\varphi(0) = 0$. If $f \in \mathcal{L}^p(X, \mu)$ with $f \geq 0$, let $A = \{\varphi(g) \mid g \in \mathcal{L}^p(X, \mu), 0 \leq g \leq f\}$ and define $\tilde{\varphi}(f) = \sup A$. Since the zero functional is in A, we have $\tilde{\varphi}(f) \geq 0$. Since $0 \leq g \leq f$, it follows that

$$\varphi(f) \leq \tilde{\varphi}(f) \leq \|\varphi\|\|f\|_p. \tag{1}$$

Now suppose that $f \in \mathcal{L}^p(X, \mu)$. Define $\phi : L^p(X, \mu) \to \mathbb{R}$ by $\phi(f) = \tilde{\varphi}(f^+) - \tilde{\varphi}(f^-)$. Note that if $f \geq 0$ then $\phi(f) = \tilde{\varphi}(f) - \tilde{\varphi}(0) = \tilde{\varphi}(f) \geq 0$, so ϕ is positive. Since $\tilde{\varphi}$ is continuous, so is ϕ. We next claim that ϕ is linear.

Let $f, h \in \mathcal{L}^p(X, \mu)$. Assume first $f, h \geq 0$. For any $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$, linearity of φ implies that

$$\tilde{\varphi}(f + h) \geq \varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2),$$

so $\tilde{\varphi}(f + h) \geq \tilde{\varphi}(f) + \tilde{\varphi}(h)$. For the converse inequality let again $\varepsilon > 0$ and choose $0 \leq g \leq f + h$ such that $\tilde{\varphi}(f + h) < \varphi(g) + \varepsilon$. Let $g_1 = \min\{g, f\}$. Since $0 \leq g_1 \leq f$, $0 \leq g - g_1$ and $g \leq f + h$ we necessarily have $0 \leq g - g_1 \leq h$ (assume not and reach a contradiction). Then

$$\tilde{\varphi}(f + h) < \varphi(g - g_1) + \varphi(g_1) + \varepsilon < \varepsilon + \tilde{\varphi}(h) + \tilde{\varphi}(f),$$

from which the converse inequality follows. Thus $\tilde{\varphi}$ is linear on positive valued functions.

For arbitrary f, h note that $f + h = (f^+ + h^+) - (f^- + h^-)$. But $f + h = (f + h)^+ - (f + h)^-$, so $(f + h)^+ + (f^- + h^-) = (f^+ + h^+) + (f + h)^-$. Applying $\tilde{\varphi}$ to both sides of the last equality and using that $\tilde{\varphi}$ is linear on positive valued functions shows that $\phi(f + h) = \phi(f) + \phi(h)$.

For homogeneity, again suppose $f \geq 0$ and let $a > 0$. Then

$$\begin{aligned}\tilde{\varphi}(af) &= \sup\{\varphi(g) \mid 0 \leq g \leq af\} = \sup\{\varphi(g) \mid 0 \leq \frac{1}{a}g \leq f\} \\ &= \sup\{\varphi(ag) \mid 0 \leq g \leq f\} = a \sup\{\varphi(g) \mid 0 \leq g \leq f\},\end{aligned}$$

using linearity of φ in the last equality. For arbitrary f and $a > 0$ note that

$$\phi(af) = \tilde{\varphi}((af)^+) - \tilde{\varphi}((af)^-) = \tilde{\varphi}(af^+) - \tilde{\varphi}(af^-) = a(\tilde{\varphi}(f^+) - \tilde{\varphi}(f^-)) = a\phi(f).$$

But $(-af)^+ = af^-$ and $(-af)^- = af^+$, so homogeneity of ϕ is true for all $a \in \mathbb{R}$.

Finally, we have $\varphi = \phi - (\varphi - \phi)$, where both ϕ and $\varphi - \phi$ are linear, positive, and bounded. □

Note that a similar proof can be found in the book "The elements of integration and Lebesgue measure" by R. G. Bartle, Wiley Classics Library (see Lemma 8.13).