**Suggested solution to the exam in MAT4410, December 17, 2012.**

**Problem 1.** Let $\lambda$ denote Lebesgue measure on $X = [0, \infty)$. Find the limit

$$\lim_{n\to\infty} \int_{X\times X} e^{-(\frac{x^2y^2}{n}+x+y)} d(\lambda \otimes \lambda)(x, y).$$

Solution: We have $e^{-(\frac{x^2y^2}{n}+x+y)} \to e^{-(x+y)}$ as $n \to \infty$ pointwise on $X \times X$. By Tonelli's theorem, which applies since all the functions involved are measurable and non-negative, it follows that

$$\int_{X\times X} e^{-(x+y)} d(\lambda \otimes \lambda)(x, y) = (\int_X e^{-x} d\lambda(x))(\int_X e^{-y} d\lambda(y)),$$

and since $\int_X e^{-x} d\lambda(x) = \lim_n \int_{[0,n]} e^{-x} dx = 1$ e.g. by Monotone Convergence theorem, it follows that $e^{-(x+y)} \in L^1(X \times X)$. Then

$$\lim_{n\to\infty} \int_{X\times X} e^{-(\frac{x^2y^2}{n}+x+y)} d(\lambda \otimes \lambda)(x, y) = 1$$

by Dominated Convergence Theorem.

**Problem 2.** Let $c_0$ denote the Banach space of sequences converging to zero. Let $a = \{a_n\}_{n\geq 1}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n b_n$ is convergent for every $\{b_n\}_{n\geq 1} \in c_0$. For every $k \geq 1$, define $T_k : c_0 \to \mathbb{C}$ by $T_k(b) = \sum_{j=1}^{k} a_j b_j$ for $b = \{b_n\}_{n\geq 1}$ in $c_0$.

2a. Show that $T_k$ is a bounded linear functional for every $k \geq 1$.

2b. Conclude that $a \in l^1(\mathbb{N})$. What is the relationship between $\|a\|_1$ and $\|T_k\|$ for $k \geq 1$?

Solution for 2a: since $b_n \to 0$, the sequence $b = \{b_n\}_{n\geq 1}$ is bounded. Then

$$|T_k(b)| \leq \sum_{j=1}^{k} |a_j b_j| \leq \|b\|_\infty \sum_{j=1}^{k} |a_j|.$$

Hence $\|T_k\| \leq \sum_{j=1}^{k} |a_j|$, so $T_k$ is bounded (you need to fill in the details for proving that $T_k$ is linear).

Solution for 2b: fix $b = \{b_n\}_{n\geq 1}$ in $c_0$. Then

$$\lim_{k\to\infty} T_k(b) = \lim_{k\to\infty} \sum_{j=1}^{k} a_j b_j \leq \sum_{j=1}^{\infty} a_j b_j < \infty.$$

Thus by the Banach-Steinhaus theorem, the map $Tb = \lim_{k\to\infty} T_k(b)$ defines a bounded operator $T : c_0 \to \mathbb{C}$. Moreover, $\|T\| = \sup_{k\ge 1} \|T_k\|$. Let $\alpha_j \in \mathbb{C}$ such that $a_j\alpha_j = |a_j|$ for every $j \ge 1$. Since

$$T_k(\alpha_1, \dots, \alpha_k, 0 \dots) = \sum_{j=1}^{k} a_j\alpha_j = \sum_{j=1}^{k} |a_j|,$$

it follows that $\|T_k\| = \sum_{j=1}^{k} |a_j|$ for $k \ge 1$. Hence $\|a\|_1 = \sum_{n=1}^{\infty} |a_n| = \|T\|$, so that $a \in l^1(\mathbb{N})$.

**Problem 3.** 4a. Formulate a consequence of the Hahn-Banach extension theorem for linear functionals appropriate for linear subspaces of normed spaces.

Solution. If $X$ is a normed space, then a bounded linear functional $l$ on a linear subspace $Y$ admits an extension $\bar{l}$ to a bounded linear functional with $\|l\| = \|\bar{l}\|$ (choose the convex function $\phi(\alpha x) = \|x\||\alpha|$ as a bound for $l$).

3b. If $X$ is a normed space and $Y \subset X$ is a linear subspace, show that $Y$ is dense in $X$ if and only if the only element $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$ is the zero functional $\varphi_0(x) = 0$ for all $x \in X$.

Solution: Suppose first $Y$ is dense. Let $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$. We must show that $\varphi(x) = 0$ for all $x$. If there is $x \in X$ such that $\varepsilon_0 = |\varphi(x)| > 0$, choose $y \in Y$ with $\|x - y\| < \varepsilon_0/\|\varphi\|$. Then $|\varphi(x)| = |\varphi(x - y)| \le \|\varphi\| \cdot \|x - y\| < \varepsilon_0$, a contradiction. Thus $\varphi = \varphi_0$. For the converse direction, if there is $x \in X \setminus \overline{Y}$, then $d = \operatorname{dist}(x, Y) > 0$, and by a consequence to the Hahn-Banach theorem for normed spaces there is $\varphi \in X^*$ with $\varphi(x) = d$ and $\varphi(y) = 0$ for all $y \in Y$. Thus $\varphi \ne \varphi_0$, a contradiction to the assumption.

3c. Suppose that $\mu$ is a Borel measure on $[0, 1]$ such that $\int_{[0,1]} x^k d\mu(x) = 0$ for all $k \ge 1$. Show that $\mu = 0$.

Solution: (Note that in the hypothesis one should assume $k \ge 0$ in order to avoid complications.) Let $X = C[0, 1]$ with the supremum norm, and let $Y$ be the subspace of polynomials in one variable. It is known that $Y$ is dense in $X$. By the Riesz Representation theorem, there is a bounded functional $\varphi(f) = \int_{[0,1]} f d\mu$ on $X$. The assumption that $\varphi(x^k) = 0$ for all $k \ge 0$ implies by continuity of $\varphi$ that $\varphi$ vanishes on all elements of $Y$. Then $\varphi$ is the zero functional by 3b, so $\mu([0, 1]) = \|\varphi\| = 0$, and therefore $\mu = 0$.

3d. (This problem is independent of problems 3b and 3c.) Let $\lambda$ be the Lebesgue measure on $[0, 1]$ and let $L^p([0, 1], \lambda)$ for $1 < p < \infty$ be the Banach space of $p$-integrable, complex-valued functions on $[0, 1]$. Suppose that $\{f_k\}_{k\ge 1}$ is a sequence of elements in $L^p([0, 1], \lambda)$ and $\{a_k\}_{k\ge 1}$ is a sequence of complex numbers for which there exists $C > 0$ such that

$$|\sum_{k=1}^{m} \alpha_k a_k| \leq C\big(\int_{[0,1]} |\sum_{k=1}^{m} \alpha_k f_k(x)|^p d\lambda(x)\big)^{1/p}$$

for any choice of complex numbers $\alpha_1, \ldots, \alpha_m$ and any $m \geq 1$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Show that there is $g \in L^q([0,1], \lambda)$ such that

$$\int_{[0,1]} f_k(x)g(x)d\lambda(x) = a_k, \ \forall k \geq 1.$$

Solution: Let $Y = \text{span}\{f_k \mid k \geq 1\}$. Define $\varphi_0 : Y \to \mathbb{C}$ by

$$\varphi_0(\sum_{k=1}^{m} \alpha_k f_k) = \sum_{k=1}^{m} \alpha_k a_k$$

for any choice of complex numbers $\alpha_1, \ldots, \alpha_m$ and any $m \geq 1$. This is a linear map, and the assumption gives that $\varphi_0$ is bounded with $\|\varphi_0\| \leq C$. By the Hahn-Banach theorem there is an extension $\varphi \in L^p([0,1])^*$. Since $L^p([0,1])^* \cong L^q([0,1])$, there is a function $g \in L^q([0,1])$ such that

$$\varphi(f) = \int_{[0,1]} fg d\lambda$$

for every $f \in L^p([0,1])$. Hence $\int_{[0,1]} f_k(x)g(x)d\lambda(x) = \varphi_0(f_k) = a_k$ for all $k \geq 1$, as claimed.

**Problem 4.** Let $(X, \Sigma)$ be a measurable space and $\lambda$, $\mu$ be measures on $\Sigma$ with $\lambda$ finite, $\mu$ $\sigma$-finite and $\lambda \ll \mu$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(A) < \delta \Rightarrow \lambda(A) < \varepsilon, \ \forall A \in \Sigma.$$

Solution: If $f = d\lambda/d\mu$, then let $X_n = \{x \in X \mid |f(x)| \leq n\}$ for every $n \geq 1$. Proceed as in the proof of Lemma 9.19 (Teschl): first $\lambda(X \setminus X_n) \to 0$, so there is $n$ such that $\lambda(X \setminus X_n) < \varepsilon/2$. Take now $\delta = \varepsilon/(2n)$.

4b. Let $X = \mathbb{R}$ with its Borel $\sigma$-algebra $\mathcal{B}$ and let $\lambda$ be Lebesgue measure on $\mathcal{B}$. Define a measure on $\mathcal{B}$ by

$$\mu(A) = \int_A \frac{1}{e^{|x|}} d\lambda(x) \text{ for all } A \in \mathcal{B}.$$

Explain why $\lambda \ll \mu$. Solution: first note that $\mu \ll \lambda$ by construction, because $1/e^{|x|}$ is measurable and non-negative. Since $h = d\mu/d\lambda$ is the function

$h(x) = e^{-|x|}$, which satisfies $h(x) > 0$ for all $x$, we have by a known result that $1/h$ is the Radon-Nikodym derivative $d\lambda/d\mu$.

4c. With $(\mathbb{R}, \mathcal{B})$, $\lambda$ and $\mu$ as in question 4b, show that the conclusion of problem 4a does not hold.

Solution: Let $\varepsilon = 1$. Let $\delta > 0$ arbitrary, and choose $n \in \mathbb{N}$ such that $1/e^n < \delta$. Let $A = [n, n+1]$. For $x \in A$, we have $e^x \geq e^n$, so $h(x) \leq 1/e^n < \delta$ on $A$. Thus

$$\mu(A) = \int_{[n,n+1]} \frac{1}{e^{|x|}} d\lambda(x) < \int_{[n,n+1]} \delta d\lambda(x) = \delta\lambda(A) = \delta,$$

but $\lambda(A) \geq 1$.