MAT 3500/4500, Mandatory assignment 2012

Turn in your paper by 14:30 on November 1st, in the box in the 7th floor corridor in the mathematics building. Remember to use the official cover page, available next to the box. To get your assignment accepted, you must have at least 50% correct answers. Partial credit can be given if parts of your arguments are correct, so always show your work.

Problem 1. In a metric space X, define the closed balls centered at x by $\overline{B}(x,r) = \{y|d(x,y) \leq r\}$. Show that the closure of a ball is contained in the closed ball,

$$\overline{B(x,r)} \subset \overline{B}(x,r).$$

Are these sets always equal ?

Problem 2. Let X be an infinite set with the cofinite (finite complement) topology and Y a Hausdorff space. Characterize the continuous functions from X to Y.

Problem 3. Show that a map $f : X \to Y$ is open if and only if $f(IntA) \subset Intf(A)$ for all subsets A of X, and closed if and only if $\overline{f(A)} \subset f(\overline{A})$.

Problem 4. Let $p : X \to Y$ and $s : Y \to X$ be continuous maps such that $p \circ s = id$. Show that p is a quotient map and s an imbedding.

Problem 5. Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. We extend f to a map between the one-point compactifications X^* and Y^* by letting $f(\infty) = \infty$. When is this extension continuous ?

Problem 6. Let A and B be compact subspaces of X and Y and W an open set in $X \times Y$ containing $A \times B$. Show that there are open sets $U \supset A$ and $V \supset B$ such that

$$A \times B \subset U \times V \subset W$$

Problem 7. We call a point x an *accumulation point* for a sequence (x_n) in a topological space X if every neighbourhood U of x contains infinitely many elements of the sequence. Show that the set of accumulation points equals $\bigcap_n F_n$ where $F_n = \overline{\{x_k | k \geq n\}}$. Use this to show that the set of accumulation points is closed and nonempty if X is compact.

Problem 8. Show that if $f : X \to Y$ is a quotient map, Y is connected and $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.

THE END