- Recall theorem from analysis: If $f_n : [a,b] \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly, then f is continuous.

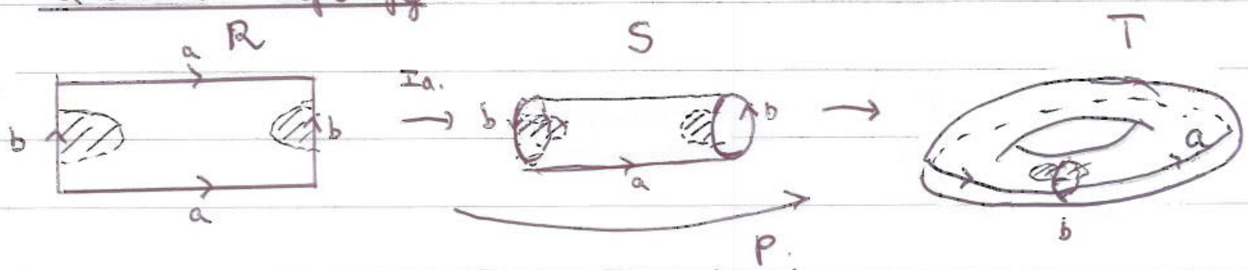
- Def. (Uniform convergence.) X topological space, Y metric space. $f_n : X \to Y$. We say that f_n converge uniformly to f if for any $\epsilon > 0$ there is an N such that $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$ when $n \geq N$.

- Th. If all f_n are continuous, then f is continuous.

Pf: Enough to prove: If $x \in X$, $\epsilon > 0$, then there is $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ when $d(x,y) < \delta$. By uniform convergence there is N s.t. $d(f_n(y), f(y)) < \frac{\epsilon}{3}$ for all $n \geq N$. Since f_N is continuous, there is $\delta > 0$ such that $d(f_N(x), f_N(y)) < \frac{\epsilon}{3}$ when $d(x,y) < \delta$. This gives

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Quotient topology.



$p : R \to T = S^1 \times S^1$. $\quad U \subset T$ open $\Leftrightarrow p^{-1}(U)$ open

- Def. Let $p : X \to Y$ be a surjective map of top. spaces. We say that p is a quotient map if $U \subset Y$ is open iff $p^{-1}(U)$ is open in X.

This is eq. to A closed in Y iff $p^{-1}(A)$ is closed in X.

Some more set theory

- Let X be a set. We shall see that the following are more or less the same thing:

1) Partitions of X in disjoint sets: $X^* = \{A_\alpha\}_{\alpha \in I}$.

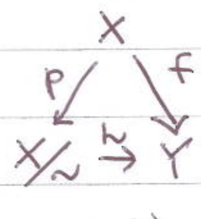
2) Equivalence relations $\sim$ on X. (ER)

3) Surjective maps $f : X \to Y$

We have already seen that 1) and 2) are the same.
If $\sim$ is an equivalence relation on X, we get a partition of X into equivalence classes

$$[x] = \{y \in X \mid y \sim x\}$$

The set of equivalence classes is denoted by $X/\sim$, called a quotient space.
The map $p : X \to X/\sim$ is surjective, showing that 2) gives rise to 3).
If we start with a surjective map $f : X \to Y$, we can define an ER on X by $x \sim y$ if $f(x) = f(y)$. It is easy to see that this is an ER and that the map $h : X/\sim \to Y$ defined by $h([x]) = f(x)$ is a well defined bijection. We have a commutative diagram



i.e. f and the quotient map p are the "same" (up to a bijection.)

- Given a surjective map $f : X \to Y$ and a subset $A \subset X$, we always have

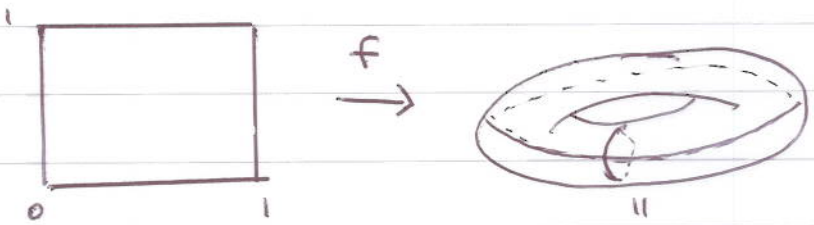
$$A \subset f^{-1}(f(A))$$

If $A = f^{-1}(f(A))$ we say that A is <u>saturated</u> (mettet). In the descriptions above this translates to:

1) If $x \in A$ is in some A_α, then $A_\alpha \subset A$
2) If $x \in A$ and $y \sim x$, then $y \in A$
3) If $x \in A$ and $f(y) = f(x)$, then $y \in A$

- We can think of a surjective map $f : X \to Y$ as "glueing together pieces of X to make Y". A piece is an equivalence class.

Glueing edges of a rectangle to make a torus, can be realized like this



$f(s,t) = (e^{2\pi i s}, e^{2\pi i t})$

$X = [0,1] \times [0,1]$ $\qquad$ $S^1 \times S^1$

What are the saturated sets?

i) Any subset of the open rectangle $(0,1) \times (0,1)$
ii) Sets $E \times \{0,1\}$, where $E \subset (0,1)$ is an arbitrary subset, i.e. any subset of the open bottom edge, together with the same set on top edge
iii) Sets $\{0,1\} \times F$, where $F \subset (0,1)$ is an arbitrary subset. Same as ii) with left/right instead of bottom/top.
iv) The set consisting of the four corner points

Finally, any union of the above sets. These are all the saturated sets.

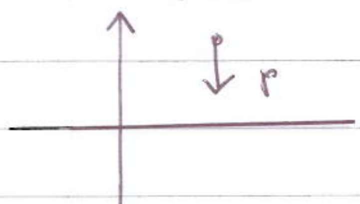
- In general $f^{-1}(f(A))$ is always saturated, the smallest saturated set containing A. It is called the saturation (mehirgen) of A.

- We return to quotient maps $p: X \to Y$. p is continuous, but this is not enough; if $V = p^{-1}(U)$ is open, i.e. V is a saturated open set, then $U = p(V)$ is open.
  This is equivalent to: If A is a saturated closed set then $p(A)$ is closed

- Corollary. If $p: X \to Y$ is surjective, continuous and open or closed, then p is a quotient map.

- These are sufficient, but not necessary conditions.
  It is easy to see that a quotient map $p: X \to Y$ is open iff the saturation of an open set is open
  p is closed iff the saturation of a closed set is closed

- Example 1



$p: \mathbb{R}^2 \to \mathbb{R}$, projection on x-axis

p is open, hence a quotient map.

p is not closed, since the curve C defined by $xy = 1$ is closed, but $p(C) = \mathbb{R} \setminus \{0\}$ is not closed. The saturation of C is $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ which is not closed.

- Example 2. $p: [0,1] \to S^1$, $p(t) = e^{2\pi i t}$ is continuous and closed (check it!), hence a quotient map. It is not open, since $[0,\frac{1}{2})$ is open in $[0,1]$ but $p([0,\frac{1}{2}))$ is not open in S^1. The saturation of $[0,\frac{1}{2})$ is $[0,\frac{1}{2}) \cup \{1\}$ which is not open.

All of these statements are also true for the map from a rectangle to the torus $p: \mathbb{R} \to T$.

- Given a topological space X and a surjective map $f: X \to Y$ to a set Y, we can define a topology on Y by
$$J = \{U \subset Y \mid f^{-1}(U) \text{ is open in } X\}.$$
This is a topology, since $\emptyset = f^{-1}(\emptyset)$, $X = f^{-1}(Y)$,
$$f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha) \text{ and } f^{-1}\left(\bigcap_{i=1}^m U_i\right) = \bigcap_{i=1}^m f^{-1}(U_i)$$

This makes f into a quotient map. The topology is the finest such that f is continuous.

- If $p: X \to Y$ is a quotient map, $A \subset X$ a subset and $B = p(A)$, is $p|_A : A \to B$ a quotient map?

In general no: In example 1, let $A = C \cup \{(0,0)\}$. Then $p: A \to \mathbb{R}$ is continuous, $\{(0,0)\}$ is a saturated open set of A, but $p(0,0) = 0$ which is not open. The problem is that A is not saturated.

- For any surjective map $f: X \to Y$ between sets, $A \subset X$, $B = f(A)$ it is true that:
If $E \subset A$ is a saturated set for f, then it is also saturated for $f|_A$. The opposite is true if A itself is a saturated set. (for f).

- <u>Theorem</u> Let $p: X \to Y$ be a quotient map, $A \subset X$ a saturated set, $B = p(A)$ and $q = p|_A : A \to B$. Then

(1) If A is either open or closed, then q is a quotient map
(2) If p is either open or closed, then q is a quotient map.

Proof: In all cases $q: A \to B$ is continuous. Hence we have to prove that if $V \subset A$ is a saturated open (or closed) subset of A then $q(V)$ is an open subset of B. "Saturated" here means for the map q, but by the previous comment V is also saturated as a subset of X, i.e. for p.

(1) If A is open and $V \subset A$ is saturated and open, then V is also saturated and open as a subset of X, i.e $p(V) = q(V)$ is open in X and hence also in B.
(2) If p is open and $V \subset A$ is open, then there is an open subset $U \subset X$ such that $V = U \cap A$. Since A is saturated, it follows that $q(V) = p(U \cap A) = p(U) \cap p(A) = p(U) \cap B$ (draw Venn diagram!) which is open in B since $p(U)$ is open in Y. Hence q is an open map and therefore a quotient map.

The closed case follows by replacing "open" by "closed" in the argument above.

- A composition of quotient maps is a quotient map.

- If $f: X \to Y$ is a map of sets, we say that $g: X \to Z$ is <u>compatible with f</u> if $f(x) = f(x') \Rightarrow g(x) = g(x')$. This is equivalent to the existence of a map $h: Y \to Z$ such that $g = h \circ f$. (We simply define h by $h(y) = g(x)$ if $y = f(x)$)

- Theorem Let $p: X \to Y$ be a quotient map and $g: X \to Z$ a map compatible with p, i.e. $g = f \circ p$ for a map $f: Y \to Z$. Then (1) f is continuous $\Leftrightarrow$ g is continuous

  (2) f is a quotient map $\Leftrightarrow$ g is a quotient map

Proof: The implications $\Rightarrow$ are already known.

$\Leftarrow$ (1). If $U \subset Z$ is open, then $g^{-1}(U) \subset X$ is open since g is continuous. But $g^{-1}(U) = p^{-1}(f^{-1}(U))$, hence $f^{-1}(U) \subset Y$ is open since p is a quotient map.

$\Leftarrow$ (2). f must be surjective since g is. Also, f is continuous by (1). We must prove that if $f^{-1}(U)$ is open, then U must be open. Again $g^{-1}(U) = p^{-1}(f^{-1}(U))$ which must be open since p is continuous. But then U is open since g is a quotient map.

- Theorem Let $g: X \to Z$ be a surjective continuous map and define an ER on X by $x \sim y$ if $g(x) = g(y)$. Then $f: X/\sim \to Z$ defined by $f([x]) = g(x)$ is bijective and continuous. ($X/\sim$ has the quotient topology). Also
  a) f is a homeomorphism $\Leftrightarrow$ g is a quotient map
  b) If Z is Hausdorff, so is $X/\sim$.

Proof: We know that f is bijective and $g = f \circ p$ where $p: X \to X/\sim$ is the quotient map, hence f is continuous by (1).

a) $\Rightarrow$ Follows since g is a composition of quotient maps.

$\Leftarrow$ From (2) we get that f is a quotient map. But a bijective quotient map is a homeomorphism

b) If $[x] \neq [y]$, then $g(x) \neq g(y)$ and have disjoint nbhs U_x and U_y in Z. But then $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are disjoint nbhs of $[x]$ and $[y]$.