STK2130 Spring 2014 – Mandatory assignment

Deadline Thursday March 20th, 14:30

You are allowed to collaborate and discuss the problems with other students, but each student has to formulate her or his own answers. You should give the names of the students you collaborate with, so that it is possible to compare the written solutions.

The assignment consists of 2 Problems over 3 pages. Make sure you have the complete assignment.

The answer to the exam project should be handed in on paper on the 7th floor in NHA (Niels Henrik Abels hus). You may deliver a handwritten or Latex/Word-processed answer to the project in English, Norwegian or any other Scandinavian language

Problem 1

A Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ is defined by the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) The chain has three classes, $\mathcal{C}_0 = \{0, 1\}, \mathcal{C}_1 = \{2, 3\}, \mathcal{C}_2 = \{4\}$ where $\mathcal{C}_0$ is transient, $\mathcal{C}_1$ is closed and $\mathcal{C}_2$ is absorbing (and closed). Explain why this is so.

Which of the classes are recurrent?

b) Let T be the time until the chain enters one of the closed classes and define $\mu_i = \mathrm{E}[T|X_0 = i]$ for $i \in \mathcal{C}_0$.

Explain why the μ_i satisfy the following two equations

$$(1) \qquad \mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5}$$

$$(2) \qquad \mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5}$$

Solve the equations (1) and (2) to obtain μ_0 and μ_1.

c) Let q_i be the probability that the chain ends up in state 4 conditional on $X_0 = i$. Find and explain equations for obtaining q_0 and q_1. Solve the equations.

d) Let s_{ij} denote the expected number of visits to states $j = 0$ and $j = 1$ conditional on $X_0 = i$. Find and solve equations for determining the s_{ij}.

Problem 2

Markov chains are used in numerous situations in real life. In physics, queueing theory, Internet, statistics, economy, social sciences, etc. In this exercise we will try to illustrate how one may use Markov chains in sociology. This exercise is inspired by the work of three economists "D. Acemoglu, G. Egorov, K. Sonin, *Political model of social evolution.* Proceedings of the National Academy of Sciences 108: 21292–21296. (2011)" (You do not need to look at this article).

The authors propose a model using Markov processes for describing the dynamics of political and social changes in a society. We will illustrate in a very simple example how this can be done.

Consider a society or population. Suppose that this society lives in a political regime among the following three: $\Omega = \{\text{Dictatorship, Elections, Democracy}\}$ which we code as $S = \{0, 1, 2\}$ being $0 =$ Dictatorship, $1 =$ Elections and $2 =$ Democracy. This society behaves as follows: While in a dictatorship the probability that a revolution breaks out is α in such a case this society calls for elections to decide. Then democracy is established if there is a majority (more than 50% in favour). The probability that the inhabitants vote NO is β. While in democracy, the probability that a *coup d'état* breaks out will be denoted by γ, being γ a small number hopefully. Once the society has reached democracy it will preferably stay there.

(a) Explain why the political status of this society can be modelled by using a Markov chain. Plot a diagram to help yourself.

(b) Explain why the transition probability matrix of such a process is

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ \beta & 0 & 1-\beta \\ \gamma & 0 & 1-\gamma \end{pmatrix}$$

and comment on the different states. Is this process ergodic? Remember that in a finite-state Markov chain all recurrent states are positive recurrent.

(c) Given that there is a current dictatorship, what is the probability that there is democracy after three periods?

(d) Assuming $\alpha, \beta, \gamma > 0$, explain why the process is irreducible and aperiodic. Show that the stationary distribution is

$$\pi_0 = \frac{\gamma}{\gamma(\alpha+1) + \alpha(1-\beta)}, \quad \pi_1 = \frac{\alpha\gamma}{\gamma(\alpha+1) + \alpha(1-\beta)}, \quad \text{and} \quad \pi_2 = \frac{\alpha(1-\beta)}{\gamma(\alpha+1) + \alpha(1-\beta)}.$$

Is it unique? Why?

(e) We say that a society is *obedient* if they tend to accept a dictatorial regime, that is, the probability of revolution goes to 0 as time goes by. On the other hand, we say that a society is *revolutionary* if the chances of a revolution breakout get closer to 1 and the probability of obtaining democracy in the elections converges towards 1 as time goes by.

Compute the stationary distributions for both an obedient and revolutionary society and comment on that.

(f) Assume there is democracy. There is a conspiracy in the air. The military is preparing a coup d'état but they still do not know who will take the command. So γ is not clearly specified. Statistical studies have revealed that γ has the following density function $f_\gamma(x) = 3(1-x)^2$ for $0 \leqslant x \leqslant 1$. Compute the probability that the coup d'état will not take place within the next n periods. That is, $P(X_n = 2 | X_0 = 2, X_1 \neq 0, \ldots, X_{n-1} \neq 0)$. What happens when n increases?