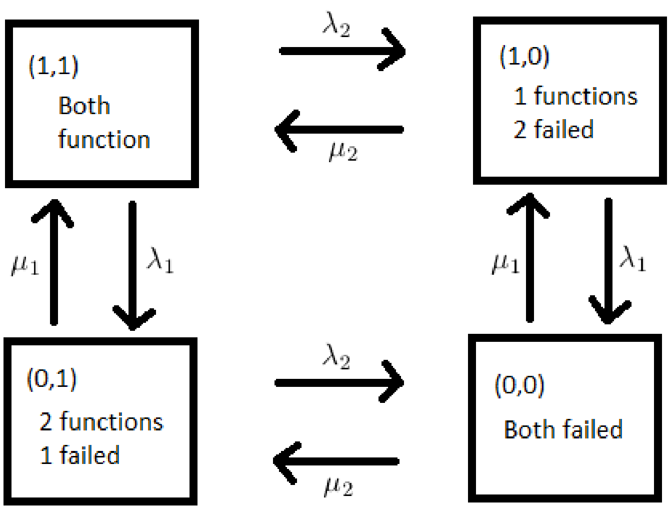
STK2130: Some of the problems for 06.05.2016

Exercise 6.10

Consider two machines. Machine i operates for an exponential time with rate λ_i and then fails; its repair time is exponential with rate μ_i, $i = 1, 2$. The machines act independently of each other. Define a four-state continuous-time Markov chain that jointly describes the condition of the two machines. USe the assumed independence to compute the transition probabilities and then verify Kolmogorov backward equations.



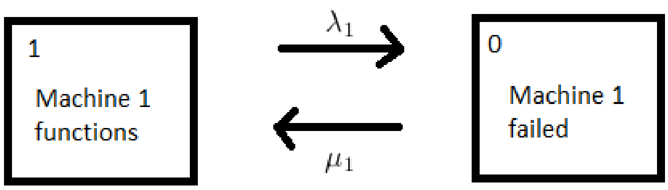
Consider the state space $S = \{(0,0), (1,0), (0,1), (1,1)\}$ and denote by $P^k_{(i,j)}$ the probability that machine $k = 1, 2$ makes a transition from i to j, where $i, j \in \{0, 1\}$ (fail/function). Then by independence

$$P_{(i,j),(k,l)} = P(X(t) = (k,l)|X(t) = (i,j)) = P^1_{(i,k)}(t)P^2_{j,l}(t)$$

for all $i, j, k, l \in \{0, 1\}$.

To compute the probabilities, let us say, $P^1_{i,k}(t)$ we consider the problem The functions for $P^1_{ik}(t)$ and $P^2_{jl}(t)$ can be found in page 388. They are found by using BKE for the single problem. We compute for instance $P^1_{00}(t)$ and P^1_{10}.

$$P^1_{00}{}'(t) = \sum_{k \neq 0} q_{0k} P_{k0}(t) - v_0 P^1_{00}(t) = \mu_1 P_{10}(t) - \mu_1 P_{00}(t)$$



$$P^1_{10}{}'(t) = \sum_{k \neq 1} q_{1k} P_{k0}(t) - v_1 P^1_{10}(t) = \lambda_1 P_{00}(t) - \lambda_1 P_{10}(t)$$

We check that Kolmogorov's backward equation holds for $P_{(0,1),(0,0)}(t)$, for instance.

$$\begin{aligned} P'_{(0,1),(0,0)}(t) &= P^1_{(0,0)}{}'(t) P^2_{10}(t) + P^1_{00}(t) P^2_{10}{}'(t) \\ &= \lambda_1 \left(P^1_{10}(t) - P^1_{00}(t) \right) P^2_{10}(t) + P^1_{00}(t) \mu_2 \left(P^2_{00}(t) - P^2_{10}(t) \right) \end{aligned}$$

On the other hand,

$$\begin{aligned} P'_{(0,1),(0,0)}(t) &= \sum_{(k,l) \neq (0,1)} q_{(0,1),(k,l)} P_{(k,l),(0,0)}(t) - v_{(0,1)} P_{(0,1),(0,0)}(t) \\ &= q_{(0,1),(0,0)} P_{(0,0),(0,0)}(t) + q_{(0,1),(1,1)} P_{(1,1),(0,0)}(t) - v_{(0,1)} P_{(0,1),(0,0)}(t) \\ &= \lambda_2 P^1_{00}(t) P^2_{00}(t) + \mu_1 P^1_{10}(t) P^2_{10}(t) - (\mu_1 + \lambda_2) P^1_{00}(t) P^2_{10}(t) \\ &= P^2_{10}(t) \mu_1 \left(P^1_{10}(t) - P^1_{00}(t) \right) P^2_{10}(t) + P^1_{00}(t) \lambda_2 \left(P^2_{00}(t) - P^2_{10}(t) \right) \end{aligned}$$

Exercise 6.8

The number of failed machines is a birth and death process with $\lambda_0 = 2\lambda$ $\lambda_1 = \lambda$ $\mu_1 = \mu_2 = \mu$ also note that $\mu_{n+1} = \lambda_n = 0, \forall n > 1$ Simply plug this into the Forward and/or Backward equations

Exam June 2005

On a very contaminated garbage dump all the rats become sterile, however, the population will not die out since there is immigration of rats from the area outside the dump.

Let $X(t)$ be the number of rats on the dump at time t. We assume that $\{X(t), t \geqslant 0\}$ is a time homogeneous Markov process with continuous time, state space $S = \{0, 1, 2, \dots\}$ and

$$p_{ji}(\Delta t) = \begin{cases} \nu \Delta t + o(\Delta t) \text{ if } j = i + 1 \\ i\mu \Delta t + o(\Delta t) \text{ if } j = i - 1 \\ 1 - (\nu + i\mu)\Delta t + o(\Delta t) \text{ if } j = i \\ o(\Delta t) \text{ if } |j - i| > 1 \end{cases}$$

where $p_{ji}(t) = P(X(t+u) = j | X(u) = i)$.

Note: First observe tat this is a birth and death process with immigration (with no births but only immigration) so $\mu_n = \mu, n \geqslant 1$ and $\lambda_n = \nu, n \geqslant 0$, so example 6.4 can be of help.

(a) Derive $q_{ii} = p'_{ii}(0)$ and $q_{ji} = p'_{ji}(0)$ for $j \neq i$. Set up the matrix $Q = (q_{ji})_{j,i \in S}$.

Simply, by definition of derivative

$$q_{ii} = p'_{ii}(0) = \lim_{\Delta t \to 0} \frac{p_{ii}(\Delta t) - p_{ii}(0)}{\Delta t} = -(\nu + i\mu).$$

For $j = i + 1$

$$q_{i+1,i} = p'_{i+1,i}(0) = \lim_{\Delta t \to 0} \frac{p_{i+1,i}(\Delta t) - p_{i+1,i}(0)}{\Delta t} = \nu.$$

For $j = i - 1$

$$q_{i-1,i} = p'_{i-1,i}(0) = \lim_{\Delta t \to 0} \frac{p_{i-1,i}(\Delta t) - p_{i-1,i}(0)}{\Delta t} = i\mu.$$

For j s.t. $|j - i| > 1$

$$q_{ji} = p'_{ji}(0) = \lim_{\Delta t \to 0} \frac{p_{ji}(\Delta t) - p_{ji}(0)}{\Delta t} = 0.$$

Hence the matrix looks like

$$Q = \begin{pmatrix} -\nu & \nu & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\nu+\mu) & \nu & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\mu & -(\nu+2\mu) & \nu & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & i\mu & -(\nu+i\mu) & \nu & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \cdots \end{pmatrix} \tag{0.1}$$

(b) Set up equations to determine the stationary distribution $\pi = (\pi_0, \pi_1, \dots)^t$, for $\{X(t), t \geqslant 0\}$. Show that

$$\pi_i = \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i \pi_0, \quad i = 1, 2, \dots$$

fit into the equations, and finally determine π.

We have to solve the system of equatins:

$$v_i P_i = \sum_{k \neq j} q_{kj} P_k$$

for all $j \in S$. A mnemonic rule for this is "outbound rates = sum of neighbours outbounds". So for each $i \geqslant 0$

$$(i\mu + \nu)P_i = \nu P_{i-1} + (i+1)\mu P_{i+1}.$$

Finally, check that $P_i = \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i P_0$ satisfy the equations.

(c) Let $X(0) = 0$ and derive a set of differential equations to determine $p_{j0}(t), j \in S$. (Hint: Use Chapman-Kolmogorov equations $p_{j0}(t+u) = \sum_{k=0}^{\infty} p_{jk}(u)p_{k0}(t)$ with $u = \Delta t$.)

$$\begin{aligned} p_{j0}(t+u) - p_{j0}(u) &= \sum_{k=0}^{\infty} p_{jk}(u)p_{k0}(t) - p_{j0}(u) \\ &= \sum_{k\neq 0} p_{jk}(u)p_{k0}(t) + p_{j0}(u)p_{00}(u) - p_{j0}(u) \\ &= \sum_{k\neq 0} p_{jk}(u)p_{k0}(t) - (1 - p_{00}(u))p_{j0}(u) \end{aligned}$$

Therefore

$$\begin{aligned} p'_{j0}(t) = \lim_{\Delta t\to 0} \frac{p_{j0}(t+\Delta t) - p_{j0}(t)}{\Delta t} &= \lim_{\Delta t\to 0} \sum_{k\neq 0} \frac{p_{jk}(\Delta t)}{\Delta t} p_{k0}(t) - \lim_{\Delta t\to 0} \frac{1 - p_{00}(\Delta t)}{\Delta t} p_{j0}(t) \\ &= \sum_{k\neq 0} q_{jk}p_{k0}(t) - \nu p_{j0}(t) \end{aligned}$$

See Lemma 6.2 as well.

(d) Show that $m(t) = E[X(t)|X(0) = 0]$ satisfies the differential equation

$$m'(t) = -\mu m(t) + \nu, \quad m(0) = 0$$

and find $m(t)$.

Write $E[X(t+\Delta t)|X(t)]$ for a small time change Δt. Then, given we know $X(t)$, at time $X(t+\Delta)$ we might have gone up by one with probability $\nu\Delta t + o(\Delta t)$, or down by one with probability $X(t)\mu\Delta t + o(\Delta t)$ or just stay with $X(t)$ individuals with the rest of probability. Hence,

$$\begin{aligned} E[X(t+\Delta t)|X(t)] &= X(t)\left(1 - (\nu + X(t)\mu)\Delta t + o(\Delta t)\right) + (X(t)+1)\left(\nu\Delta t + o(\Delta t)\right) \\ &\quad + (X(t)-1)\left(X(t)\mu\Delta t + o(\Delta t)\right) \\ &= X(t) + \nu\Delta t - \mu X(t)\Delta t + C \cdot o(\Delta). \end{aligned}$$

Apply $E[\cdot]$ and then

$$\frac{E[X(t+\Delta t)] - E[X(t)]}{\Delta t} = \nu - \mu E[X(t)]$$

which gives

$$m'(t) = \nu - \mu m(t)$$

and since $X(0) = 0$ we have $m(0) = 0$.

(1) We solve the homogeneous eq. $m'(t) + \mu m(t) = 0$ (the characteristic poly. is $x + \mu = 0$ with root $x = -\mu$, so $m_h(t) = e^{-\mu t}$. A particular solution should be a polynomial since the independent term is a polynomial, so $m_p(t) \equiv C$ (because the independent term is of degree 0). So

$$m_p'(t) + \mu m_p(t) = 0 + \mu C = \nu \Rightarrow C = \nu/\mu$$

so $m_p(t) = \frac{\nu}{\mu}$.

Finally, $m(t) = K m_h(t) + m_p(t)$ with $0 = m(0) = K + \frac{\nu}{\mu} = 0 \Rightarrow K = -\frac{\nu}{\mu}$. Hence

$$m(t) = -\frac{\nu}{\mu}\left(e^{-\mu t} - 1\right).$$

(2) If we define $h(t) := -\mu m(t) + \nu$ then $h'(t) = -\mu(-\mu m(t) + \nu) = -\mu h(t)$ with $h(0) = -\mu m(0) + \nu = \nu$. So

$$h(t) = Ce^{-\mu t},\ h(0) = \nu \Rightarrow h(t) = \nu e^{-\mu t}.$$

Hence

$$m(t) = \frac{\nu - h(t)}{\mu} = \frac{\nu}{\mu}(1 - e^{-\mu t}).$$

Note on Erlang Distribution and its CDF

Please see https://en.wikipedia.org/wiki/Erlang_distribution to discover that Erlang is a particular case of Gamma when the shape is a natural number (positive integer).