STK2130: Rest of the problems for 13.05.2016

Exercise 10.7

Compute the expression for $P\left(\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\right)$.

Let $Y_{t_1,t_2}(s,x) := \{\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\} \in \mathcal{F}$. Then we condition on $X(t_1)$ and observe that

$$P(Y_{t_1,t_2}(s,x)) = \int_{-\infty}^{\infty} P(Y_{t_1,t_2}(s,x)|X(t_1) = y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)} dy.$$

Furthermore, observe that

$$P(Y_{t_1,t_2}(s,x)|X(t_1) = y) = 1, \ y \geqslant x$$

and for $y \leqslant x$

$$P(Y_{t_1,t_2}(s,x)|X(t_1) = y) = P\left(\max_{0 \leqslant s \leqslant t_2 - t_1} B(s) > x - y\right) = 2P(X(t_2 - t_1) > x - y)$$

Exercise 10.9

Let $\{X(t), t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the joint density function of $X(s)$ and $X(t)$, $s \leqslant t$?

(1) A Brownian motion $X(t)$ with drift coefficient μ and variance parameter σ^2 has mean μt and variance $\sigma^2 t$. So, the density function of $X(t)$ is

$$f_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}.$$

So the joint density of $(X(s), X(t))$ is a two-dimensional Gaussian distribution with mean

$$E[(X(s), X(t))^*] = (\mu s, \mu t)^*$$

($*$ means transpose) and covariance matrix

$$E[(X(s) - \mu s, X(t) - \mu t)^*(X(s) - \mu s, X(t) - \mu t)] = \begin{pmatrix} \sigma^2 s & \sigma^2 s \\ \sigma^2 s & \sigma^2 t \end{pmatrix}.$$

Above we used at some point that $E[X(s)X(t)] = \sigma^2(s \wedge t) = \sigma^2 s$ since $s \leqslant t$.

(2) We know $X(t) = \mu t + \sigma B(t)$ where $B(t)$ is a standard Brownian motion. Then

$$(X(s), X(t)) = (\mu s, \mu t) + \sigma(B(s), B(t))$$

Since the joint density of $(B(s), B(t))$ is a two-dimensional normal distribution with mean $(0, 0)^*$ and covariance

$$\begin{pmatrix} s & s \\ s & t \end{pmatrix}$$

then $(X(s), X(t))$ has the two-dimensional Gaussian distribution with mean $(\mu s, \mu t)^*$ and covariance

$$\sigma^2 \begin{pmatrix} s & s \\ s & t \end{pmatrix}.$$

Notice that $Cov(B(s), B(t)) = E[(B(s) - E[B(s)])(B(t) - E[B(t)])] = E[(B(s) - 0)(B(t) - 0)] = = E[(B(s))(B(t))] = E[(B(t) - B(s) + B(s))(B(s))] = E[B^2(s) + (B(s) - B(t)))(B(s) - B(0))] = E[B^2(s)] + E[(B(s) - B(t)))(B(s) - B(0)] = E[B^2(s)] = Var[B(s)] = s$