STK2130: Rest of the problems for 15.04.2016

Exercise 6.5

There are N individuals in a population, some of whom have a certain infection that spreads as follows. Contacts between two members of this population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and a noninfected individual, then with probability p the noninfected individual becomes infected. Once infected, an individual remains infected throughout. Let $X(t)$ denote the number of infected individuals of the population at time t.

(a) Is $\{X(t), t \geqslant 0\}$ a continuous-time Markov chain? Yes, future events or outcomes depend only on the current state. In other words, the Markov property is fulfilled. See page 372.

(b) Specify its type. It's a birth process. Starting with $X(0)$ (infected) individuals then as time goes by, there become more and more infected individuals with no possibility of recovering. Thus $X(t)$ increases. Here, being infected is as they are born (they get multiplied). So $\mu_n = 0$ and X is a pure birth process.

(c) Starting with a single infected individual, what is the expected time until all members are infected?

The following is how to compute the $(N \times 1) \times (N \times 1)$ transition matrix which is not part of the exercise:

Imagine two boxes, the first one with n infected individuals at time t, i.e. $X(t) = n$ and the other one with noninfected individuals, i.e. $N - n$. At a Poisson rate λ we take out two (no matter from which box) and if they are both sane, they remain sane. If they are both infected, they remain infected. If one is infected and the other is not, then with prob. p it gets infected. So, $X(t) = n$ then

$$X(t+h) = n \text{ if we extract two infected or two noninfected.}$$

$$X(t+h) = n+1 \text{ if we extract one of each and } *p$$

The probability of extracting two sane ones is:

$$P(\text{ two sane}) = \left(\frac{N-n}{N}\right)^2$$

The probability of extracting two infected ones is:

$$P(\text{ two infected}) = \left(\frac{n}{N}\right)^2$$

The probability of extracting one of each is:

$$P(\text{one of each}) = 2\frac{n}{N}\frac{N-n}{N}.$$

So the transition probabilities are:

$$p_{i,j} = \delta_{i=j=0}$$

$$p_{n,n} = \frac{(N-n)^2 + n^2}{N^2} + 2\frac{n}{N}\frac{N-n}{N}(1-p)$$

$$p_{n,n+1} = 2\frac{n}{N}\frac{N-n}{N}p.$$

Observe that $p_{n,n} + p_{n,n+1} = 1$ for all $n \geqslant 1$.

The birth transition rates are then

$$\lambda_n = \lambda\frac{n(N-n)}{\binom{N}{2}}p, \; n = 0, \ldots, N-1$$

i.e. rate of encounter λ times the probability that one infected meets a noninfected and the noninfected becomes infected.

We know that the time between changes of state are exponentially distributed with rate λ_n, thus mean $\frac{1}{\lambda_n}$, then starting from $X(0) = 1$, until all are infected, that is $X(t) = N$, we need to sum up:

$$\text{Total mean time until infection } = \sum_{n=1}^{N-1}\frac{1}{\lambda_n} = \frac{\binom{N}{2}}{\lambda p}\sum_{n=1}^{N-1}\frac{1}{n(N-n)}.$$