STK2130: Rest of the problems for 22.04.2016

Exercise 6.11 (d,e)

(d)

$$\begin{aligned}P_{1j}(t) &= P(X(t)=j|X(0)=1) = P(X(t)\geqslant j|X(0)=1) - P(X(t)\geqslant j+1|X(0)=1)\\ &= P(T_1+\dots+T_{j-1}\leqslant t) - P(T_1+\dots T_j\leqslant t)\\ &= e^{-\lambda t}(1-e^{-\lambda t})^{j-1}.\end{aligned}$$

We see directly that $X(t)=j|X(0)=1 \sim \text{Geom}(e^{-\lambda t})$ which is also natural.

(e) The sum of i independent geometrics, each having parameter $p=e^{-\lambda t}$ is a negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual, i.e.

$$(X(t)=j|X(0)=i) = \sum_{k=1}^{i}(X(t)=j|X(0)=1) \sim \sum_{k=1}^{i}\text{Geom}(e^{-\lambda t}) = i\text{Geom}(e^{-\lambda t}) = \text{NB}(i, e^{-\lambda t})$$

Exam June 2006

(a) Obviously by the statement of the problem: we have two machines that are always occupied when the number of customers is greater or equal than two and only one machine is occupied when there is one customer in the system, correspondingly no service takes place when there are no customers in the system.

(b) Derive $q_{ii}=p'_{ii}(0)$ (or not define the diagonal at all) and $q_{ji}=p'_{ji}(0)$ for $j\neq i$. Set up the matrix $Q=(q_{ji})_{j,i\in S}$.

Simply, by definition of derivative

$$q_{ii}=p'_{ii}(0)=\lim_{\Delta t\to 0}\frac{p_{ii}(\Delta t)-p_{ii}(0)}{\Delta t} = -(\mu_i+\lambda_i).$$

For $j=i+1$

$$q_{i+1,i}=p'_{i+1,i}(0)=\lim_{\Delta t\to 0}\frac{p_{i+1,i}(\Delta t)-p_{i+1,i}(0)}{\Delta t} = \mu_i.$$

For $j=i-1$

$$q_{i-1,i}=p'_{i-1,i}(0)=\lim_{\Delta t\to 0}\frac{p_{i-1,i}(\Delta t)-p_{i-1,i}(0)}{\Delta t} = \lambda_i.$$

For j s.t. $|j-i|>1$

$$q_{ji}=p'_{ji}(0)=\lim_{\Delta t\to 0}\frac{p_{ji}(\Delta t)-p_{ji}(0)}{\Delta t} = 0.$$

Hence plugging in the parameters found in (a) we find that the matrix looks like

$$Q = \begin{pmatrix} -(\lambda) & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \cdots \end{pmatrix} \tag{0.1}$$