STK2130: Rest of the problems for 26.02.2016

Exercise 4.52

Taxi driver's problem. Let X_n the Markov process of "zone of the n-th trip". Then $X_n \in S = \{A, B\}$ with transition probability matrix

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}.$$

Denote by Y_n the profit of the n-th trip, so we wish to compute $E[Y_n]$. We need some more information, whether we are going to change zone in the next trip or not and from where, so we need the current zone and the past one in order to know whether we changed or stayed. Thus,

$$E[Y_n] = E\left[E[Y_n|X_{n-1}, X_n]\right] = \sum_{i,j \in S} E[Y_n|X_{n-1} = i, X_n = j]P(X_{n-1} = i, X_n = j).$$

Expanding the above expression

$$\begin{aligned}
E[Y_n] &= E[Y_n|X_{n-1} = A, X_n = A]P(X_{n-1} = A, X_n = A) \\
&+ E[Y_n|X_{n-1} = A, X_n = B]P(X_{n-1} = A, X_n = B) \\
&+ E[Y_n|X_{n-1} = B, X_n = A]P(X_{n-1} = B, X_n = A) \\
&+ E[Y_n|X_{n-1} = B, X_n = B]P(X_{n-1} = B, X_n = B) \\
&= 6P(X_{n-1} = A, X_n = A) \\
&+ 12P(X_{n-1} = B, X_n = A) \\
&+ 12P(X_{n-1} = A, X_n = B) \\
&+ 8P(X_{n-1} = B, X_n = B) \\
&= 6p_{AA}P(X_{n-1} = A) \\
&+ 12p_{BA}P(X_{n-1} = B) \\
&+ 12p_{AB}P(X_{n-1} = A) \\
&+ 8p_{BB}P(X_{n-1} = B)
\end{aligned}$$

In the long run, ($\pi_A = 3/7$, $\pi_B = 4/7$)

$$\lim_n E[Y_n] = (6p_{AA} + 12p_{AB})\pi_A + (12p_{BA} + 8p_{BB})\pi_B = 8.857$$

Exercise 4.53 (in a bit more detail)

Find the average premium received per policyholder of the insurance company Example 4.27 if $\lambda = 1/4$ for one-third of its clients, and $\lambda = 1/2$ for two-thirds of its clients.

Since expectation is linear, we can compute the average preimoum for policy holders of respective $lambda$'s and take the weighted mean. For $\lambda = 1/4$ we have $a_0 = 0.7788$, $a_1 = 0.1947$, $a_2 = 0.0243$ and the transition probability matrix is:

$$\begin{pmatrix} 0.7788 & 0.1947 & 0.0243 & 0.0022 \\ 0.7788 & 0 & 0.1947 & 0.0265 \\ 0 & 0.7788 & 0 & 0.2212 \\ 0 & 0 & 0.7788 & 0.2212 \end{pmatrix}$$

which has stionary probabilities:

$$\pi_1 = 0.693 \quad \pi_2 = 0.197 \quad \pi_3 = 0.079 \quad \pi_4 = 0.031$$

so the average premium:

$$\text{Average}_{1/4} = \pi_1 200 + \pi_2 250 + \pi_3 400 + \pi_4 600 = 238,21.$$

For $\lambda = 1/2$ it is done in Example 4.27 and is

$$\text{Average}_{1/2} = 326,275.$$

Hence,

$$\text{Average} = \frac{1}{3}238,21 + \frac{2}{3}326.275 = 296.99 \approx 297.$$

Exercise 4.64

Consider a branching process (forgreningsprosess) having $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in this population is given by $\frac{1}{1-\mu}$. What if $X_0 = n$?

See page 246-247. If $\mu < 1$ we know that the population will die out. This can be proved but also seems reasonable. So this gives a hint that the sum of $E[X_n]$ for all $n \geqslant 0$ should be finite. Indeed, the exepected number of individuals is given by

$$E[X_n] = \mu^n$$

So,

$$\sum_{t \geqslant 0} E[X_t] = \sum_{t \geqslant 0} \mu^t = \frac{1}{1-\mu}$$

since $\mu < 1$ and the series is an infinite geometric sum.

We have the relation $E[X_t] = \mu E[X_{t-1}] = \cdots = \mu^t E[X_0] = n\mu^t$. Therefore

$$\sum_{t \geqslant 0} E[X_t] = \sum_{t \geqslant 0} n\mu^t = \frac{n}{1-\mu}$$

Exercise 4.66

For a branching process calculate π_0 when

(a) $P_0 = 1/4$, $P_2 = 3/4$. Since $\mu = \frac{3}{2} > 1$ we use equation (4.20) from the book page 248.

$$\pi_0 = \frac{1}{4} + \frac{3}{4}\pi_0^2 \iff \pi_0 \in \left\{1, \frac{1}{3}\right\} \Rightarrow \pi_0 = 1/3.$$

(b) $P_0 = 1/4$, $P_1 = 1/2$, $P_2 = 1/4$.

$$\pi_0 = \frac{1}{4} + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0^2 \iff (\pi_0 - 1)^2 = 0 \Rightarrow \pi_0 = 1.$$

Again we could have skipped this since $\mu = \sum_{j \geqslant 0} jP_j = \frac{1}{2} + 2\frac{1}{4} = 1 \leqslant 1$, so $\pi_0 = 1$.

(c) $P_0 = 1/6$, $P_1 = 1/2$, $P_3 = 1/3$. Here, $\mu = 1/2 + 1 = 3/2 \geqslant 1$. So,

$$\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3 \iff \pi_0 \in \left\{1, \frac{-1 \pm \sqrt{3}}{2}\right\} \Rightarrow \pi_0 = \frac{-1 + \sqrt{3}}{2} \approx 0.366.$$