STK2130: Rest of the problems for 29.01.2016

Exercise 4.4

Consider a process $\{X_n, n \geqslant 0\}$ which takes on the values 0,1 or 2. Suppose

$$P(X_{n+1} = j | X_n = i) = \begin{cases} P_{ij}^I, & n \text{ even}, \\ P_{ij}^{II}, & n \text{ odd} \end{cases}$$

where $\sum_{j=0}^{2} P_{ij}^I = \sum_{j=0}^{2} P_{ij}^{II} = 1$ for all $i = 0, 1, 2$. Is $\{X_n, n \geqslant 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

The process $\{X_n, n \geqslant 0\}$ is a Markov chain, but not time-homogeneous, since the transition probabilities depend on "time" n. Nevertheless, by enlarging the state space we may transform our Markov process into a time-homogeneous process so that, for example, the Chapman-Kolmogorov equation is fulfilled. Take $S = \{01, 2, \overline{0}, \overline{1}, \overline{2}\}$ and define a new process Y_n such that

$$Y_n = i \iff X_n = i \text{ and } n \text{ even}$$

and

$$Y_n = \overline{i} \iff X_n = i \text{ and } n \text{ odd}.$$

Then we can write a transition probability matrix (enlarged) $P = (P_{ij})_{i,j \in S}$

$$P = \left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array} \right)$$

where $A = (P_{ij}^I)_{ij \in \{0,1,2\}}$ and $B = (P_{ij}^{II})_{ij \in \{0,1,2\}}$.

Exercise 4.13

Let P be the transition probability of a Markov chain. Argue that if for some positive integer r, P^r has all positive entries (is regular) then so does P^{n+r} for all $n \geqslant r$.

Let $A =: P^r = (a_{ij})_{i,j \in S}$ and $P = (p_{ij})_{i,j \in S}$. Then

$$P^{r+1} = AP = \left(\sum_{k \in S} a_{ik} p_{kj} \right)_{i,j \in S}.$$

Now, fix an arbitrary element of the matrix P^{r+1}, row i and column j. We have $a_{ik} > 0$ for all $k \in S$ so $\sum_{k \in S} a_{ik} p_{kj} = 0$ if, and only if $p_{kj} = 0$ for all $k \in S$. In such a case, P^r could never have positive entries for any $r \geqslant 0$.