29.01.2015 | Ch 4: 2, 4, 5, 6, 7, 13

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ex 4.2) $X_t$, $t \geq 1$ $X_t \in \{0, 1\}$

$p(X_{t+n} | X_0, \dots, X_t, \dots, X_{t+n-1}) \neq p(X_{t+n} | X_{t+n-1})$

$y_t$, $t \geq 3$ $y_t \in \{\underline{\varepsilon_1}, \underline{\varepsilon_2}, \underline{\varepsilon_3}\}$, $\varepsilon_i \in \{0, 1\} \forall i = \overline{1,3}$

| | 000 | 100 | 010 | 001 | 110 | 101 | 011 | 111 |
|---|---|---|---|---|---|---|---|---|
| 000 | $p_{000,000}$ | 0 | 0 | $p_{000,001}$ | 0 | 0 | 0 | 0 |
| 100 | | | | | | | | |
| 010 | | | | | | | | |
| 001 | | | | | | | | |
| 110 | | | | | | | | |
| 101 | | | | | | | | |
| 011 | | | | | | | | |
| 111 | 0 | | | | $p_{111,110}$ | | 0 | $p_{111,111}$ |

ex 4.5) $X_n$, $n \geq 0$, $X_n \in \{0, 1, 2\}$ (2)

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

$P(X_0 = 0) = \pi_0 = \frac{1}{4}$

$P(X_0 = 1) = \pi_1 = \frac{1}{4}$

$P(X_0 = 2) = \frac{1}{2} = \pi_2$

$E\{X_3\}$ - ?

double expectation formula

$$E\{X\} = E\{E\{X|y\}\}$$

$$E\{E\{X|y\}\} = \int_{\Omega_y} p(y) E\{X|y\} \, dy =$$

$$= \int_{\Omega_y} p(y) \int_{\Omega_x} x \, p(X|y) \, dx \, dy =$$

$$= [\text{Bayes formula}] = \int_{\Omega_y} \cancel{p(y)} \int_{\Omega_x} x \frac{p(x,y)}{\cancel{p(y)}} \, dx \, dy =$$

$$= [\text{change the order of integration}] =$$

$$= \int_{\Omega_x} x \underbrace{\int_{\Omega_y} p(x,y) \, dy}_{p(x)} \, dx = \int_{\Omega_x} x \, p(x) \, dx = E\{X\}$$

$$E\{X_3\} = E\{E\{X_3 | X_0\}\} =$$

$$\sum_{i=0}^{2} E(X_3 | X_0 = i) \, P(X_0 = i) =$$

$$= \sum_{i=0}^{2} P(X_0 = i) * \left( \sum_{j=0}^{2} j \, P(X_3 = j | X_0 = i) \right) =$$

$$= \boxed{\sum_{i=0}^{2} \sum_{j=0}^{2} j \, P(X_0 = i) \, P_{ij}^{(3)}} =$$

$$= \left[ P^{(3)} \begin{pmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{26} \end{pmatrix} \right] = 0.9814$$

whilst:

$$\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 2 = 1\frac{1}{4}$$

for initial states

ex 4.6. 3

$$P=\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \quad P^{(n)} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{pmatrix}$$

proof by induction

1) $n=1$. $$P^{(1)} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1) & \frac{1}{2}-\frac{1}{2}(2p-1) \\ \frac{1}{2}-\frac{1}{2}(2p-1) & \frac{1}{2}+\frac{1}{2}(2p-1) \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$ – true

2) consider it is true for $n=t$ and see that the induction holds for $n=t+1$

$$P^{(t+1)} = P^{(t)} \cdot P = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^t & \frac{1}{2}-\frac{1}{2}(2p-1)^t \\ \frac{1}{2}-\frac{1}{2}(2p-1)^t & \frac{1}{2}+\frac{1}{2}(2p-1)^t \end{pmatrix}$$

$$\times \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \Big[ P^{(t+1)}_{11} = \left(\frac{1}{2}+\frac{1}{2}(2p-1)^t\right)p + \left(\frac{1}{2}-\frac{1}{2}(2p-1)^t\right)(1-p) =$$

$$= \cancel{\tfrac{1}{2}p} + \frac{1}{2}(2p-1)^t p + \frac{1}{2}(1-\cancel{p}) - \frac{1}{2}(2p-1)^t(1-p) =$$

$$= \frac{1}{2}\left((2p-1)^t(p+p-1)+1\right) = \underbrace{\frac{1}{2}(2p-1)^{t+1}+\frac{1}{2}}$$

the same for $P^{t+1}_{1,2}$ $\Big] =$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{t+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{t+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{t+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{t+1} \end{pmatrix}$$, true

the statement is proved by induction!

ex 4.7

$X_t$ - Markov chain $\in \{0, 1, 2, 3\}$

0 - rained today & yest.

1 - rained today, not yest.

2 - rained yest., not today.

3 - didn't rain neither yest. nor today

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$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.7 & 0 & 0.3 & 0 \\ 1 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0 & 0.4 & 0 & 0.6 \\ 3 & 0 & 0.2 & 0 & 0.8 \end{array}$$

$P(X_{t+2} = \{0 \text{ or } 1\} \mid X_t = 3) = P^{(2)}_{30} + P^{(2)}_{31} =$

$$= \left[ P^{(2)} = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.2 & 0.15 & 0.3 \\ 0.2 & 0.12 & 0.2 & 0.48 \\ 0.1 & 0.16 & 0.2 & 0.64 \end{pmatrix} \right] = 0.26$$