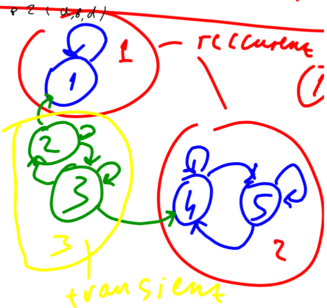
12.02.2016 / Ch 4: 23, 30, Exam 2012: 2(a, b, d)

a) $S = \{1, 2, 3, 4, 5\}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

$$\sum_{n \geq 0} p_{ii}^{(n)} = +\infty$$



b) starting in state 2: $X_0 = 2$,

N_j : $N_j = \min_n \{X_n = j, X_{n+1} = j, \dots, X_\infty = j\}$

$(P(N_1 < +\infty \mid X_0 = 1))$

$P(N_j < +\infty \mid X_0 = j) = 1$

$P(N_j < +\infty \mid X_0 = i$, i is absorbing, $i \neq j) = 0$

$P(N_j < +\infty \mid X_0 = i$, i - transient, com. with $j) =$

$= q_i^{(j)}$

$P(N_j < +\infty \mid X_0 = i) = \left[P(X) = \int_{\Omega_y} P(X, y)\, dy \right.$ add one, and integrate it out $\left.\right] =$

$= \sum_{k \in S} P(N_j < +\infty, X_1 = k \mid X_0 = i) = [$ Bayes theorem

$= \sum_{k \in S} P(N_j < +\infty \mid X_1 = k, \cancel{X_0 = i})\, P(X_1 = k \mid X_0 = i)$ because of Markov main property

$= \sum_{k \in S} \underbrace{P(N_j < +\infty \mid X_1 = k)}_{= q_k^{(j)}}\, P(X_1 = k \mid X_0 = i) =$

$= \sum_{k \in S} q_k^{(j)} p_{ik}$ define

$P(N_1 < +\infty \mid X_0 = 2) = q_2^{(1)} = \sum_{k \in S} q_k^{(1)} \cdot p_{2k} =$

$= q_1^{(1)} p_{21} + q_2^{(1)} p_{2,2} + q_3^{(1)} p_{2,3} + q_4^{(1)} p_{2,4} + q_5^{(1)} p$

$q_1^{(1)} = 1$, $q_4^{(1)} = 0$, $q_5^{(1)} = 0$

$q_3^{(1)} = q_2^{(1)} p_{3,2} + q_3^{(1)} p_{3,3} + p_{31}$ - if we begin in state (3)

$$\begin{cases} q_2^{(1)} = q_2^{(1)} P_{22} + q_3^{(1)} P_{23} + P_{21} \cdot 1 \\ q_3^{(1)} = q_2^{(1)} P_{32} + q_3^{(1)} P_{33} + P_{31} \cdot 1 \end{cases}$$, P_{ij} are known

$\Rightarrow$ $q_2^{(1)} = 0.5161$

$q_3^{(1)} = 0.32258$

d)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$ $\Rightarrow$ states 4, 5 are in a closed class $\Rightarrow$ create an own Markov chain

$$Q = \begin{array}{c} 4 \\ 5 \end{array} \begin{pmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{pmatrix} \Rightarrow \Pi = \Pi Q \Rightarrow$$

Π - limiting probabilities

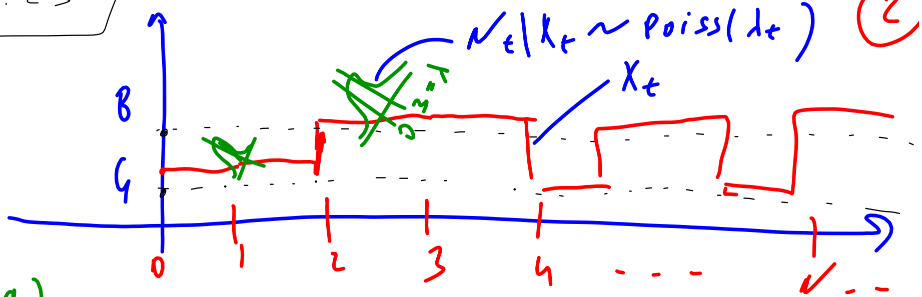
$\Pi_4 = P_{44} \Pi_4 + \Pi_5 P_{54}$

$\Pi_5 = P_{45} \Pi_4 + P_{55} \Pi_5$ $\iff$ $\Pi_4 = \frac{45}{99}$, $\Pi_5 = \frac{54}{99}$

4.30 is exactly the same

4.2.3

2



a) $X_0 = 4$, Find $E\{N_1 + N_2\} =$

$= E\{N_1\} + E\{N_2\} =$

$= E\{\underbrace{E\{N_1 | X_1\}}_{= \begin{cases} 1, & X_1 = 4 \\ 3, & X_1 = 8 \end{cases}}\} + E\{\underbrace{E\{N_2 | X_2\}}_{= \begin{cases} 1, & X_2 = 4 \\ 3, & X_2 = 8 \end{cases}}\}$

$E\{E\{N_1 | X_1\}\} = \sum_{i \in S} P(X_1 = i) \cdot \lambda_i =$

$= [P(X_1 = 4) \cdot 1 + P(X_1 = 8) \cdot 3] =$

$= [P(X_1 = i) \overset{[P(x) = \int_{\eta_y} p(x,y)dy]}{=} \sum_{j \in S} P(X_1 = i, X_0 = j)$ Bayes theorem

$= \sum_{j \in S} P(X_1 = i | X_0 = j) P(X_0 = j) \Rightarrow P(X_1 = 4) = P_{44} P(X_0 = 4) +$

$+ P_{84} P(X_0 = 8) = 1 \cdot \frac{1}{2}$, $P(X_1 = 8) = P_{48} P(X_0 = 4) +$

$+ P_{88} \cdot 1 \cdot P(X_0 = 8) = \frac{1}{2}$ $] = \frac{1}{2} + \frac{3}{2} = 2$

$E\{N_2\} = E\{E\{N_2 | X_2\}\} =$

$\sum_{i \in S} P(X_2 = i) \cdot \lambda_i \overset{[P(x) = \int_{\eta_y} p(x,y)dy]}{=} \sum_{i \in S} \sum_{j \in S} P(X_2 = i, X_0 = j) \lambda_i \overset{[\text{Bayes}]}{=}$ Bayes formula

$= \sum_{i \in S} \lambda_i \sum_{j \in S} \underbrace{P(X_2 = i | X_0 = j)}_{= P^2_{ji}} P(X_0 = j) = \frac{13}{6}$

Then $\Rightarrow E(N_1 + N_2) = \frac{13}{6} + 2 = \frac{25}{6}$

c) $\lim_{t \to \infty} E\{N_t\} = \lim_{t \to \infty} E\{E\{N_t | X_t\}\} =$

$= \lim_{t \to \infty} \sum_{i \in S} P(X_t = i) \lambda_i \overset{[\lim \Sigma = \Sigma \lim]}{=} \sum_{i \in S} \lambda_i \lim_{t \to \infty} P(X_t = i) =$

$= \sum_{i \in S} \lambda_i \pi_i$; $\pi = \pi P$