19.02.2016 | ch 4: 20, 59, exam 2012 - 4(c), exam 2011

ex 4.20

P - is doubly stochastic $\Leftrightarrow \sum_{i=0}^{M} P_{ij} = 1 \quad \forall j \in \{0, \dots, M\}$

Markov chain defined by P is irreducible & a-periodic => ergodic => it has a single stationary distribution, thus we check is $\pi_j = \frac{1}{M+1}$ is this unique solution:

$$\pi_j = \sum_{i=0}^{M} P_{ij} \pi_i = \left[\pi_i = \frac{1}{M+1} \; \forall i \in \{0, \dots, M\}\right] =$$

$$= \sum_{i=0}^{M} P_{ij} \frac{1}{M+1} = \frac{1}{M+1} \sum_{i=0}^{M} P_{ij} = \frac{1}{M+1} = \pi_j$$

N/B: the problem is a little lemma

ex 4.59; $\zeta_i = \min_t \{X_t = 0 \,||\, X_t = N \mid X_0 = i\}$.

$M_i = E\{\zeta_i\}$, $M_0 = 0$, $M_N = 0$

$M_i = 1 + p M_{i+1} + q M_{i-1}$

$E\{E\{\zeta_i \mid X_1\}\} = E\{\zeta_i \mid X_1 = i+1\} p +$

$+ E\{\zeta_i \mid X_1 = i-1\} q = \left[\begin{array}{l} E\{\zeta_i \mid X_1 = i+1\} = E\{\zeta_{i+1}\} + 1 \\ E\{\zeta_i \mid X_1 = i-1\} = 1 + E\{\zeta_{i-1}\} \end{array}\right]$

$= p(\underbrace{E\{\zeta_{i+1}\}}_{= M_{i+1}} + 1) + q(\underbrace{E\{\zeta_{i-1}\}}_{= M_{i-1}} + 1) =$

$= p + q + p M_{i+1} + q M_{i-1} = [p + q = 1] =$

$1 + p M_{i+1} + q M_{i-1} = M_i$

exam 2012. Problem 2-c

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$ — P_t

$$N_2 = \min_t \left\{ X_t = 1 \text{ or } X_t = 4 \text{ or } X_t = 5 \;\middle|\; X_0 = 2 \right\}$$

$S = (I - P_t)^{-1}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, S_{ij} are times we are spending in j, given that we begin in i

$$S = \begin{pmatrix} 2.5806 & 1.6129 \\ 1.6129 & 2.2581 \end{pmatrix} \Rightarrow$$

$\Rightarrow E\{N_2\} = S_{22} + S_{23} =$

$= 1.6129 + 2.2581 = 4.1935$

$N_i = \min_t \{ X_t = 1 \text{ or } X_t = 4 \text{ or } X_t = 5 \mid X_0 = i, \text{ \& } i\text{-transient} \}$

$E\{N_i\} = \mu_i$

$E\{N_2\} = E\{E\{N_2 \mid X_1\}\} =$

$= \mu_2 = p_{22} E\{E\{N_2 \mid X_1 = 2\}\} + p_{23} E\{E\{N_2 \mid X_1 = 3\}\}$

$+ p_{21} E\{E\{N_2 \mid X_1 = 1\}\} + p_{24} E\{E\{N_2 \mid X_1 = 4\}\}$

$+ p_{25} E\{E\{N_2 \mid X_1 = 5\}\} =$

$= \Big[\, E\{E\{N_2 \mid X_1 = 2\}\} = 1 + E\{N_2\}$

$E\{E\{N_2 \mid X_1 = 3\}\} = 1 + E\{N_3\}$

$E\{E\{N_2 \mid X_1 = 1\}\} = 1$

$E\{E\{N_2 \mid X_1 = 4\}\} = 1$

$E\{E\{N_2 \mid X_1 = 5\}\} = 1 \,\Big] =$

$= p_{21} \cdot 1 + p_{24} \cdot 1 + p_{25} \cdot 1 + p_{22}(1 + E\{N_2\}) +$

$+ p_{23}(1 + E\{N_3\}) = p_{21} + \dots + p_{25} +$

$+ p_{22} E\{N_2\} + p_{23} E\{N_3\} = 1 + p_{22} E\{N_2\} + p_{23} E\{N_3\}$

$= 1 + p_{22}\mu_2 + p_{23}\mu_3$

exactly the same for $E\{N_3\} = \mu_3 \Rightarrow$

$\Rightarrow$ we get the system of equations:

$$\begin{cases} \mu_2 = 1 + p_{22}\mu_2 + p_{23}\mu_3 \\ \mu_3 = 1 + p_{32}\mu_2 + p_{33}\mu_3 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \mu_2 = 4.1935$

$\mu_3 = 3.871$

two tricks $P(X) = \int_{\Omega_y} P(X, y)\, dy =$ [Bayes formula]

$= \int_{\Omega_y} P(y \mid X) P(X)\, dy = P(X) \int_{\Omega_y} P(y \mid X)\, dy =$ — extend the space of events

$= P(X) \cdot 1$

second trick — apply double Expectation

$E\{E\{X \mid y\}\} = E\{X\}$

$x + y - y$

$x \cdot \frac{y}{y}$ — it's just algebra

exam 2015 (f)

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

starting in 1, find the exp. of time until entering states 2, 3 or 4

$\Rightarrow P_t = p \Leftrightarrow (I - P_t) = 1 - p \Leftrightarrow$

$S = \frac{1}{1-p}$ — Expectation of a geometrically distributed r.v.

$N_1 \sim \text{Geometric}\{p\}$

$E\{N_1\} = E\{\text{Geometric}\{p\}\} =$

$= \frac{1}{1-p}$