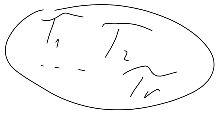
26.02.2016 | Ch4: 46, 52, 53, 60, 64, 66



a) $X_t \in S$, $S = \{0, \dots, r\}$ - # at home

P_{ij} - are transitions for X_t.

$P(X_{t+1} = 1 \mid X_t = 0) = p$

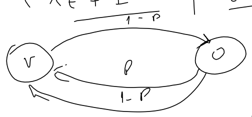
$P(X_{t+1} = 0 \mid X_t = 0) = (1-p)$.

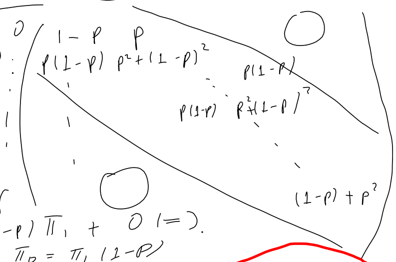
$P(X_{t+1} = r \mid X_t = r) = p \cdot p + (1-p)$

$P(X_{t+1} = j \mid X_t = i) = [|i-j| > 1] = 0$

$P(X_{t+1} = j \mid X_t = i) = [|j-i| = 1] = p(1-p)$

$P(X_{t+1} = i \mid X_t = i) = [i \neq 0, i \neq r] = p \cdot p + (1-p)^2$





$\pi = \pi P$

B) $\Downarrow$

$\pi_0 = (1-p)\pi_0 + p(1-p)\pi_1 + 0$ (=)

(=) $\pi_1 = \frac{\pi_0}{1-p}$ => $\pi_0 = \pi_1(1-p)$

$\pi_1 = p\pi_0 + (p^2 + (1-p)^2)\pi_1 + p(1-p)\pi_2$ (=)

$\pi_1 = p\pi_1(1-p) + (p^2 + (1-p^2))\pi_1 + p(1-p)\pi_2$

$\pi_1 = p\pi_1 - \cancel{p^2\pi_1} + \cancel{p^2\pi_1} + \pi_1 + p^2\pi_1 - 2p\pi_1 + p(1-p)\pi_2$

$0 = \pi_1 p(p-1) + p(1-p)\pi_2$ (=)

$\pi_1 p(1-p) = p(1-p)\pi_2$ (=> $[p \neq 0, p \neq 1]$) (=)

$\pi_1 = \pi_2$

$\pi_i = \pi_{i-1} p(1-p) + p(1-p)\pi_{i+1} + (p^2 + (1-p)^2)\pi_i =$

(=) by induction $\pi_{i-1} = \pi_i$ (=)

$\pi_i = \pi_i (p)(1-p) + p(1-p)\pi_{i+1} + (p^2 + (1-p)^2)\pi_i$

$\Longleftrightarrow$

$p(1-p)\pi_i = p(1-p)\pi_{i+1}$ (=) $\pi_i = \pi_{i+1}$ That's true $\forall i$ by induction

$\pi_r = \pi_{r-1} = \dots = \pi_1 = \frac{\pi_0}{1-p}$.

$\sum_{i=0}^{r} \pi_i = 1$ (=) $r\frac{\pi_0}{1-p} + \pi_0 = 1$ (=)

$\pi_0 = \frac{1-p}{r+1-p} = [q = 1-p] = \frac{q}{r+q}$

$\pi_i = \frac{1}{r+q} = \frac{1-p}{(r+1-p)(1-p)} = \frac{1}{r+1-p} = \frac{1}{r+q}$ $\forall i = \overline{1,r}$

c)

$\lim_{t\to\infty}(P(\text{he gets wet})) = \pi_0 \cdot p + \pi_r (1-p) \cdot p =$

$= \frac{1-p}{r+1-p} \cdot p + \frac{1}{r+1-p}(1-p)p = \frac{2p(1-p)}{r+1-p}$

d) $r = 3$, find p that maximizes the fraction of time he gets wet

=> find $f(p) = \frac{2p(1-p)}{4-p}$

$f'(p) = 0$ w.r.t. p

check $f''(p^*) < 0$

$p^* = 4 \pm 2\sqrt{3}$ => $p^* = 4 - 2\sqrt{3}$

$f''(p^*) = 2\frac{-4\sqrt{3}\ 2\sqrt{3}}{(2\sqrt{3})^3} < 0$ =>

p^* – local maximum (global on $p \in (0,1]$)

4.60

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{pmatrix}$$

$\vec{\rho} = (1, 0, 0, 0) \Leftrightarrow P(X_0 = 1) = 1$

a) find Pr that 3 is entered before 4

$N_{ij} = \min_{t, \tau, \tau < t} (X_t = j, X_\tau = i)$

$P(N_{34} < +\infty \mid X_0 = 1) = \left[P(X) = \int_{\Omega_y} p(x, y)\, dy \right] =$

$= \sum_{k=1}^{4} P(N_{34} < +\infty, X_1 = k \mid X_0 = 1) = [\text{Bayes formula}] =$

$= \sum_{k=1}^{4} P(N_{34} < +\infty \mid \cancel{X_0 = 1}, X_1 = k) P(X_1 = k \mid X_0 = 1)$

$= \sum_{k=1}^{4} P(N_{34} < +\infty \mid X_1 = k) P_{1k} =$

$= [P(N_{34} < +\infty \mid X_0 = i) =: q_i] =$

$\sum_{k=1}^{4} q_k P_{1k} = 0.4 \cdot q_1 + 0.3 \cdot q_2 + 0.2 \cdot 1 + 0.1 \cdot 0 =$

$= P(N_{34} < +\infty \mid X_0 = 1) = q_1$

$\begin{cases} q_1 = 0.4 \cdot q_1 + 0.3 q_2 \\ \text{for the second equation expand } P(N_{34} < +\infty \mid X_0 = 2) \\ q_2 = 0.2 q_1 + 0.2 \cdot q_2 + 0.2 \cdot 1 + 0.4 \cdot 0 \end{cases}$

$\Rightarrow q_1 = \frac{11}{21}$

B) $E\{N_{34} < +\infty \mid X_0 = 1\}$ then

$E\{E\{N_{34} < +\infty \mid X_0 = 1; X_1\}\} =$

$= [E\{N_{34} < +\infty \mid X_0 = i\} = M_i] =$

$= \sum_{k=1}^{4} E\{N_{34} < +\infty \mid \cancel{X_0 = 1}, X_1 = k\} P_{1k} =$

$= \sum_{k=1}^{4} M_k \cdot P_{1k} = 1 + 0.4 M_1 + 0.3 M_2 + 0 + 0$

$\begin{cases} M_2 = 1 + 0.2 M_1 + 0.2 M_2 + 0 + 0 \\ M_1 = 1 + 0.4 M_1 + 0.3 M_2 \end{cases} \Rightarrow$

$M_1 = \frac{55}{21}$

4.53 $\lambda = \lambda^* \Rightarrow E\{T^*\}$ as in example 4.27

see 4.27

if $\lambda \sim f(\lambda)$.

$E(T) = E\{E\{T \mid \lambda\}\} =$

$= \sum_{k \in S_\lambda} E\{T \mid \lambda = k\} P(\lambda = k) =$

$= E(T \mid \lambda = \frac{1}{4}) \cdot \frac{1}{3} + E\{T \mid \lambda = \frac{1}{2}) \cdot \frac{2}{3} =$

$= 326.275 \cdot \frac{2}{3} + 238.21 \cdot \frac{1}{3} = 296.99$