05.11.2015 / Exercises 24, 27, then exercise 22 in r

$$\underset{n\times 1}{Y} \sim N_n(\underset{n\times p}{X}\underset{p\times 1}{\beta}, \sigma^2 \underset{n\times n}{V_0})$$ — fixed. and known.

a)

$A_{n\times(n-p)}$ $\quad A^T X = 0, \quad p < n$

$$\underset{n-p\times n\times n\times 1}{A^T Y} \sim \underset{n-p\times 1}{N_{n-p}}(\vec{\mu}, \Sigma_{new})$$

$$E\{A^T Y\} = A^T E\{Y\} = A^T E\{X\beta\} =$$

$$= A^T X \beta = \vec{0}$$

$$cov(A^T Y) = A^T cov(Y) A = A^T \sigma^2 V_0 A =$$

$$= \sigma^2 \underbrace{A^T V_0 A}_{V_1} = \sigma^2 V_{1\,(n-p)\times(n-p)}$$

$$z = A^T Y \sim N_{n-p}(\vec{0}, \sigma^2 V_1)$$

b)

$$f(A^T Y) = (2\pi)^{-\frac{n-p}{2}} |\sigma^2 V_1|^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(A^T Y - 0)^T (\sigma^2 V_1)^{-1} (A^T Y - 0)\right) =$$

$$= (2\pi)^{-\frac{n-p}{2}} |\sigma^2 V_1|^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2\sigma^2}(Y^T A) V_1^{-1} (A^T Y)\right).$$

$$\log f(A^T Y) = const. - \tfrac{1}{2}\log|\sigma^2 V_1| -$$

$$-\frac{1}{2\sigma^2} Y^T A V_1^{-1} A^T Y. = const - \tfrac{1}{2}\log \sigma^{2(n-p)}|V_1| - \tfrac{1}{2\sigma^2} Y^T A V_1^{-1} A^T Y =$$

$$= const^* - \frac{n-p}{2}\log\sigma^2 - \frac{1}{2\sigma^2} Y^T A V_1^{-1} A^T Y$$

$$\frac{\partial \log f(\cdot)}{\partial \sigma^2} = -\frac{n-p}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} Y^T A V_1^{-1} A^T Y = 0$$

$$-\frac{(n-p)}{2} + \frac{1}{2\sigma^2} Y^T A V_1^{-1} A^T Y = 0 \Leftrightarrow$$

$$\hat{\sigma}^2_{REML} = \left(\frac{1}{n-p} Y^T A V_1^{-1} A^T Y\right)$$

c)

$$\underbrace{Y^T A (A^T V_0 A)^{-1} A^T Y}_{scalar} = tr\left((A^T V_0 A)^{-1} A^T Y (A^T Y)^T\right)$$

$$\underset{1\times n}{Y^T} \underset{n\times n-p}{A} (\underset{n-p\times n}{A^T} \underset{n\times n}{V_0} \underset{n\times n-p}{A})^{-1} \underset{n-p\times n}{A^T} \underset{n\times 1}{Y} = \Big[[1\times n-p]\times[n-p\times n-p][n-p\times 1]\Big] =$$

$$= [1\times 1 - constant] = [tr(scalar) = scalar] =$$

$$tr(\underbrace{Y^T A}_{M_1} \underbrace{(A^T V_0 A)^{-1} A^T Y}_{M_2}) = [tr(M_1 \times M_2) = tr[M_2 \times M_1]] =$$

$$= tr\left((A^T V_0 A)^{-1} A^T Y \underbrace{Y^T A}_{(A^T Y)^T}\right) = tr\Big((A^T V_0 A)^{-1} \times$$

$$\times (A^T Y)(A^T Y)^T\Big)$$

d)

$$E\{A^T Y (A^T Y)^T\} = \sigma^2 A^T V_0 A$$

$$E\{(A^T Y - 0)(A^T Y - 0)^T\} =$$

$$cov(A^T Y) = \sigma^2 A^T V_0 A$$

$$E\{\hat{\sigma}^2_{REML}\} = E\left\{\frac{1}{n-p} Y^T A [A^T V_0 A]^{-1} A^T Y\right\} =$$

$$= \frac{1}{n-p} E\{tr(A^T V_0 A)^{-1} A^T Y (A^T Y)^T\} =$$

$$= \frac{1}{n-p} tr\left(E\{(A^T V_0 A)^{-1} A^T Y (A^T Y)^T\}\right) =$$

$$= \frac{1}{n-p} tr\left\{(A^T V_0 A)^{-1} E\{A^T Y (A^T Y)^T\}\right\} =$$

$$= \frac{1}{n-p} tr\left\{(A^T V_0 A)^{-1} \cdot \sigma^2 \underset{n-p\times n\times n\times n\times n-p}{A^T V_0 A}\right\} =$$

$$= \frac{1}{n-p} tr(\sigma^2 I_{n-p\times n-p}) = \frac{1}{n-p} \sum_{i=1}^{n-p} \sigma^2 \cdot 1 = \frac{\sigma^2 (n-p)}{n-p} =$$

$$= \sigma^2$$

e) $\tilde{A}_{n\times n-p} = A_{n\times n-p} (B)_{n-p\times n-p}$

$$\hat{\sigma}^2_{REML} = \frac{1}{n-p} Y^T A I (A^T V_0 A)^{-1} I A^T Y =$$

$$= \frac{1}{n-p} Y^T A (BB^{-1})(A^T V_0 A)^{-1} (B^T)^{-1} B^T A^T Y =$$

$$= \frac{1}{n-p} Y^T A B (B^T A^T V_0 A B)^{-1} B^T A^T Y =$$

$$= \frac{1}{n-p} Y^T A B ((AB)^T V_0 AB)^{-1} (AB)^T Y =$$

$$= [\tilde{A} = AB] = \frac{1}{n-p} Y^T \tilde{A} (\tilde{A}^T V_0 \tilde{A})^{-1} \tilde{A} Y$$

e) $X_{n\times 1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ $\sigma^2 V_{0\,n\times n} = \sigma^2 I_{n\times n} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \\ 0 & & 1 \end{pmatrix}$

$$A^T_{n-1\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ - & - & & - & & - \\ 0 & & \cdots & & 1 & -1 \end{pmatrix}$$

$$A^T X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ - & & - & & - & \\ 0 & & \cdots & & 1 & -1 \end{pmatrix}_{n-1\times n} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}_{n\times 1} = \begin{pmatrix} 1-1 \\ 1-1 \\ \\ 1-1 \end{pmatrix}_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{n-1}$$

$$\hat{\sigma}^2_{REML} = \frac{1}{n-1} Y^T A (A^T \underset{I}{V_0} A)^{-1} A^T Y =$$

$$= \frac{1}{n-1} Y^T A (A^T A)^{-1} A^T Y = [A^T A = I + \mathbb{1}\mathbb{1}^T] =$$

$$= \frac{1}{n-1} Y^T A (I + \mathbb{1}\mathbb{1}^T)^{-1} A^T Y = [\text{second hint}] =$$

$$= \frac{1}{n-1} Y^T A (I - K \mathbb{1}\mathbb{1}^T) A^T Y = \left[K = \frac{1}{\dim(\mathbb{1}^T\mathbb{1})+1} = \frac{1}{n-1+1} = \frac{1}{n} \right] =$$

$$= \frac{1}{n-1} \left(Y_1 - Y_n, Y_2 - Y_n, \ldots, Y_n - Y_n \right)_{1\times n} \begin{pmatrix} 1-\frac{1}{n} & & -\frac{1}{n} \\ & \ddots & \\ -\frac{1}{n} & & 1-\frac{1}{n} \end{pmatrix}$$

$$[Y^T A = (A^T Y)^T]$$

$$\times A^T Y = \left[s_i = (Y_i - Y_n)\left(1 - \frac{1}{n}\right) + \sum_{j\neq i} (Y_j - Y_n)\left(-\frac{1}{n}\right) = \right.$$

$$\left. = \left(Y_i - Y_n - \sum_j \frac{Y_j - Y_n}{n} = Y_i - \cancel{Y_n} - \bar{Y} + \cancel{Y_n} = Y_i - \bar{Y} \right) \right] =$$

$$\frac{1}{n-1} \left(Y_1 - \bar{Y}, \ldots, Y_n - \bar{Y} \right) \begin{pmatrix} Y_1 - Y_n \\ \vdots \\ \vdots \\ Y_n - Y_n \end{pmatrix} =$$

$$\frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})(Y_i - Y_n) =$$

$$= \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})(Y_i - \bar{Y} + \bar{Y} - Y_n) =$$

$$= \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 + \frac{1}{n-1} \underbrace{\sum_{i=1}^n (Y_i - \bar{Y})}_{0} (\bar{Y} - Y_n) =$$

$$= \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

$Y_{ij} \sim N(\mu_i, \sigma^2)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$
independent.

$$L(\mu_i, \sigma^2) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(Y_{ij} - \mu_i)^2\right)$$

$$l(\mu_i, \sigma^2) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\log \frac{1}{\sqrt{2\pi}} - \frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(Y_{ij} - \mu_i)^2\right)$$

$$\frac{\partial l(\cdot)}{\partial \mu_i} = \sum_{j=1}^{m} -\frac{1}{2\sigma^2} 2 (Y_{ij} - \mu_i)(-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow \cancel{\frac{1}{\sigma^2}}\left(\sum_{j=1}^{m} Y_{ij} - m\mu_i\right) = 0$$

$$\boxed{\hat{\mu}_i = \frac{1}{m}\sum_{j=1}^{m} Y_{ij} = \bar{Y}_{i.}} \quad \forall i \in \overline{1, n}$$

$$\frac{\partial l(\cdot)}{\partial \sigma^2} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(-\frac{1}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}(Y_{ij} - \mu_i)^2\right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \sum_{j=1}^{m} \left(-\frac{1}{2} + \frac{1}{2\sigma^2}(Y_{ij} - \mu_i)^2\right) = 0.$$

$$n m \sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij} - \mu_i)^2 = [\mu_i = \hat{\mu}_i] =$$

$$\hat{\sigma}^2_{ML} = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij} - \hat{\mu}_i)^2$$

$$E\{\hat{\sigma}^2_{ML}\} = E\left\{\frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij} - \mu_i)^2\right\} =$$

$$= \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} E\left\{(Y_{ij} - \mu_i)^2\right\} = \left[\begin{matrix} Z_{ij} = Y_{ij} - \mu_i \\ Z_{ij} \sim N(0, \sigma^2) \end{matrix}\right] =$$

$$= \frac{1}{nm} \sum_{i,j} E\{Z^2\} = \frac{1}{nm} \sum_{i,j} (Var(Z) + E^2(Z)) =$$

$$= \frac{1}{nm} \sum_{i,j} \sigma^2 = \sigma^2,$$ providing that we μ_i

now address the case when μ_i are estimated.

$$E(\hat{\sigma}^2) = \frac{1}{nm} \sum_{ij} \left(E(Y_{ij}^2) - E(\hat{\mu}_i^2)\right) = \frac{1}{nm} \sum_{ij} \left(\sigma^2 + \mu_i^2 - \frac{\sigma^2}{m} - \mu_i^2\right) =$$

$$\hat{\mu}_i \sim N\left(\mu_i, \frac{\sigma^2}{m}\right)$$

$$= \frac{1}{nm} \sum_{ij} \sigma^2 \left(1 - \frac{1}{m}\right)$$, so if μ_i are unknown and are estimated we get biased estimators for $\hat{\sigma}^2$