Solution to exercises 1.3.2006

STK 4060

Oil & Energy

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Exercise 2.21

Observed X_1, X_2, X_4, X_5 from MA(1): $X_t = Z_t + \theta Z_{t-1}$, $\{Z_t\}$ WN$(0,\sigma^2)$

$$\Rightarrow \gamma(0) = (1+\theta^2)\sigma^2, \quad \gamma(1) = \theta\sigma^2$$

a. $\hat{X}_3 = \phi_{21} X_2 + \phi_{22} X_1$

$$\mathbf{R}_2 \boldsymbol{\varphi}_2 = \boldsymbol{\rho}_2 \Rightarrow \begin{pmatrix} 1 & \dfrac{\theta}{1+\theta^2} \\ \dfrac{\theta}{1+\theta^2} & 1 \end{pmatrix} \begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{\theta}{1+\theta^2} \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} \phi_{21} = \dfrac{\theta(1+\theta^2)}{1+\theta^2+\theta^4} \\ \phi_{22} = \dfrac{-\theta^2}{1+\theta^2+\theta^4} \end{array}$$

$$v_2 = \gamma(0) - \phi_{21}\gamma(1) = \frac{(1+\theta^2)(1+\theta^4)}{1+\theta^2+\theta^4}\sigma^2$$

b. $\hat{X}'_3 = \alpha_{21} X_4 + \alpha_{22} X_5$

Cov. matrix $\{X_4, X_5\}$ = Cov. matrix $\{X_1, X_2\}$ $\Rightarrow$ $\alpha_{2k} = \phi_{2k}$

Cov. X_3 and $\{X_4, X_5\}$ = Cov. X_3 and $\{X_1, X_2\}$ $\quad k=1,2$

Same MSE, given by v_2.

Exercise 2.21 cont'd.

c. $\tilde{X}_3 = a_1 X_1 + a_2 X_2 + a_4 X_4 + a_5 X_5$

Straightforward to write down and solve the **Ra**=ρ equations. However, in an MA(1) $\{X_1, X_2\}$ and $\{X_4, X_5\}$ are independent. Thus the equations for the as become the same as in parts a. and b., and we get:

$$\tilde{X}_3 = \phi_{22} X_1 + \phi_{21} X_2 + \phi_{21} X_4 + \phi_{22} X_5, \quad \text{with } \phi_{21}, \phi_{22} \text{ as in a. and b.}$$

$$\text{MSE}: v = \gamma(0) - \mathbf{a}'\boldsymbol{\gamma} = \sigma^2 (1+\theta^2)(1 - \frac{2\theta^2}{1+\theta^2+\theta^4}) = \frac{1+\theta^6}{1+\theta^2+\theta^4}\sigma^2$$

$$\text{From a. and b.}: \; v_2 = \frac{(1+\theta^2)(1+\theta^4)}{1+\theta^2+\theta^4}\sigma^2$$

$$\text{Rel. improvement}: \; \frac{v_2 - v}{v_2} = \frac{\theta^2}{1+\theta^4}$$

Theta	Impr.
0.00	0.000
0.25	0.062
0.50	0.235
0.75	0.427
1.00	0.500

Exercise 2.22

Observed X_1, X_2, X_4, X_5 from AR(1): $X_t = \phi X_{t-1}+Z_t$, $\{Z_t\}$ WN$(0,\sigma^2) \Rightarrow \gamma(j) = \sigma^2\phi^j/(1-\phi^2)$

a. $\hat{X}_3 = \phi_{21}X_2 + \phi_{22}X_1$

$$\mathbf{R}_2\boldsymbol{\varphi}_2 = \boldsymbol{\rho}_2 \Rightarrow \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \phi \\ \phi^2 \end{pmatrix} \qquad \Rightarrow \begin{matrix} \phi_{21} = \phi \\ \phi_{22} = 0 \end{matrix}$$

$$\hat{X}_3 = \phi X_2 \qquad v_2 = \gamma(0) - \phi_{21}\gamma(1) = \sigma^2$$

b. $\hat{X}_3' = \alpha_{21}X_4 + \alpha_{22}X_5$

Same argument as in 2.21 b. Thus

$\hat{X}_3' = \phi X_4$ with MSE = σ^2

Exercise 2.22 cont'd.

c. $\tilde{X}_3 = a_1 X_1 + a_2 X_2 + a_4 X_4 + a_5 X_5$

In an AR(1) {X_1, X_2 } and {X_4, X_5 } are no longer independent. **Ra=ρ** becomes:

$$\begin{pmatrix} 1 & \phi & \phi^3 & \phi^4 \\ \phi & 1 & \phi^2 & \phi^3 \\ \phi^3 & \phi^2 & 1 & \phi \\ \phi^4 & \phi^3 & \phi & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} \phi^2 \\ \phi \\ \phi \\ \phi^2 \end{pmatrix} \Rightarrow a_1 = a_5 = 0, a_2 = a_4 = \frac{\phi}{1+\phi^2}$$

$$\text{MSE}: v = \gamma(0) - \mathbf{a}'\boldsymbol{\gamma} = \frac{\sigma^2}{1-\phi^2}\left(1 - \frac{2\phi^2}{1+\phi^2}\right) = \frac{\sigma^2}{1+\phi^2}$$

From a. and b.: $v_2 = \sigma^2$

Rel. improvement: $\dfrac{v_2\text{-}v}{v_2} = \dfrac{\phi^2}{1+\phi^4}$, same relationship as for MA(1)

Exercise 3.12

PACF for an MA(1) process. **Ra=ρ** becomes:

$$\begin{pmatrix} 1 & \frac{\theta}{1+\theta^2} & 0 & 0 & 0 \\ \frac{\theta}{1+\theta^2} & 1 & \frac{\theta}{1+\theta^2} & 0 & 0 \\ 0 & \frac{\theta}{1+\theta^2} & 1 & \ddots & 0 \\ 0 & 0 & \frac{\theta}{1+\theta^2} & \ddots & \frac{\theta}{1+\theta^2} \\ 0 & 0 & 0 & \ddots & 1 \end{pmatrix} \begin{pmatrix} \phi_{1n} \\ \phi_{2n} \\ \phi_{3n} \\ \vdots \\ \phi_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\theta}{1+\theta^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\frac{\theta}{1+\theta^2}\phi_{i-1,n} + \phi_{i,n} + \frac{\theta}{1+\theta^2}\phi_{i+1,n} = 0, \quad i = 2,\ldots,n-1$$

$$\text{Difference equation with char. eq.}: \frac{\theta}{1+\theta^2}x^2 + x + \frac{\theta}{1+\theta^2} = 0 \Rightarrow x = \begin{cases} -\theta \\ -1/\theta \end{cases}$$

$$\phi_{in} = (-1)^i(\alpha\theta^i + \beta\theta^{-i}) \qquad (1)$$

Exercise 3.12 cont'd.

$$\begin{pmatrix} 1 & \frac{\theta}{1+\theta^2} & 0 & 0 & 0 \\ \frac{\theta}{1+\theta^2} & 1 & \frac{\theta}{1+\theta^2} & 0 & 0 \\ 0 & \frac{\theta}{1+\theta^2} & 1 & \ddots & 0 \\ 0 & 0 & \frac{\theta}{1+\theta^2} & \ddots & \frac{\theta}{1+\theta^2} \\ 0 & 0 & 0 & \ddots & 1 \end{pmatrix} \begin{pmatrix} \phi_{1n} \\ \phi_{2n} \\ \phi_{3n} \\ \vdots \\ \phi_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\theta}{1+\theta^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\phi_{in} = (-1)^i (\alpha\theta^i + \beta\theta^{-i}) \qquad (1)$$

Inserting (1) into the 1. and nth line and solving for α and β give, after some manipulations:

$$\alpha = \frac{-1}{1-\theta^{2n+2}}, \quad \beta = \frac{\theta^{2n+2}}{1-\theta^{2n+2}}$$

$$\Downarrow$$

$$\phi_{in} = -\frac{(-\theta)^i}{1-\theta^{2n+2}}(1-\theta^{2(n+1-i)}), i = 1,2,\dots,n$$

$$\phi_{nn} = -\frac{(-\theta)^n}{1-\theta^{2n+2}}(1-\theta^2) = -\frac{(-\theta)^n}{\frac{1-\theta^{2n+2}}{1-\theta^2}} = -\frac{(-\theta)^n}{1+\theta^2+\theta^4+\cdots+\theta^{2n}}$$