Solution to exercises 3.5.2006

STK 4060

Oil & Energy

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Exercise 8.2

Random walk: $Y_t = M_t + W_t, \quad W_t \sim \text{WN}(0, \sigma_W^2)$ Obs. eq.

$M_{t+1} = M_t + V_t, \quad V_t \sim \text{WN}(0, \sigma_V^2)$ State eq.

(1-B)Y_t stationary with mean 0 and ACF:

$$\rho(h) = \begin{cases} 1 & h = 0 \\ \dfrac{-\sigma_W^2}{2\sigma_W^2 + \sigma_V^2} & |h| = 1 \\ 0 & |h| > 1 \end{cases} \quad \Rightarrow \quad \text{MA(1), i.e. } (1-B)Y_t = (1+\theta B)Z_t$$

(From Example 8.2.1)

In an MA(1): $\rho(1) = \dfrac{\theta}{1+\theta^2}$, so that $\dfrac{\theta}{1+\theta^2} = \dfrac{-\sigma_W^2}{2\sigma_W^2 + \sigma_V^2}$

Shall show: $\theta = -1$ iff $\sigma_V^2 = 0$, i.e. the level is constant.

$$\theta = -1 \Leftrightarrow \rho(1) = -\tfrac{1}{2} \Leftrightarrow \frac{-\sigma_W^2}{2\sigma_W^2 + \sigma_V^2} = -\tfrac{1}{2} \Leftrightarrow \sigma_V^2 = 0$$ **QED**

Exercise 8.5

Local linear trend: $Y_t = M_t + W_t, \qquad W_t \ \text{WN}(0, \sigma_W^2)$ Obs. eq.

$$\left.\begin{aligned} M_{t+1} &= M_t + B_t + V_t, & V_t \ \text{WN}(0,\sigma_V^2) \\ B_{t+1} &= B_t + U_t, & U_t \ \text{WN}(0,\sigma_U^2) \end{aligned}\right\} \text{State eq.}$$

$$(1-B)Y_t = \underbrace{M_t - M_{t-1}} + W_t - W_{t-1}$$

$$= B_{t-1} + V_{t-1} + W_t - W_{t-1}$$

$$(1-B)^2 Y_t = \underbrace{B_{t-1} - B_{t-2}} + V_{t-1} - V_{t-2} + W_t - 2W_{t-1} + W_{t-2}$$

$$= U_{t-2} + V_{t-1} - V_{t-2} + W_t - 2W_{t-1} + W_{t-2}$$

Right hand side linear combination of WN terms, thus $(1\text{-}B)^2 Y_t$ stationary.

Further, $\text{Cov}((1-B)^2 Y_t, (1-B)^2 Y_{t+h}) = 0$ for $|h| > 2$

$\Rightarrow (1-B)^2 Y_t$ 2 - correlated $\Rightarrow$ MA(2)

Exercise 8.5 cont'd.

$$(1-B)^2 Y_t = \underbrace{B_{t-1} - B_{t-2}} + V_{t-1} - V_{t-2} + W_t - 2W_{t-1} + W_{t-2}$$
$$= \quad U_{t-2} \quad + V_{t-1} - V_{t-2} + W_t - 2W_{t-1} + W_{t-2}$$

Shall show that this MA(2) process is non-invertible if σ_U=0.

$$\sigma_U^2 = 0 \Leftrightarrow U_t = 0 \;\forall\; t$$

$$\begin{aligned}(1-B)^2 Y_t &= V_{t-1} - V_{t-2} + W_t - 2W_{t-1} + W_{t-2} \\ &= (1-B)V_{t-1} + (1-B)^2 W_t \\ &= (1-B)\underbrace{\left[V_{t-1} + (1-B)W_t\right]} \\ &= \qquad (1-B)(1+\theta Z_t)\end{aligned}$$

Thus the MA side has one root B=1, i.e. non-invertible.

Exercise 8.6 a

Example 8.2.2,

Pure seasonal model: $Y_{t+1} = -Y_t - Y_{t-1} - \cdots - Y_{t-d+2} + S_t$

$$\begin{aligned} Y_{t+1} - Y_{t+1-d} &= \\ &\underbrace{- Y_t - Y_{t-1} - \cdots - Y_{t-d+2} - Y_{t-d+1}} + S_t \\ &= \qquad\qquad - S_{t-1} \qquad\qquad + S_t \\ &= (1 - B)S_t \end{aligned}$$

or

$$(1 + B + B^2 + \cdots B^{d-1})Y_{t+1} = S_t$$

$$\frac{1 - B^d}{1 - B} Y_{t+1} = S_t$$

$$(1 - B^d)Y_{t+1} = (1 - B)S_t$$

1-correlated, i.e. MA(1); non-invertible

Exercise 8.6 b

Example 8.2.3, Seasonal model with local trend:

$Y_t = Y_{Tt} + Y_{St}$, where Y_{Tt} is the local trend model from Ex. 8.5 and Y_{St} is the pure seasonal model from Ex. 8.6a

From Ex. 8.5 : $(1-B)^2 Y_{Tt} = (1+\theta_1 B + \theta_2 B^2) Z_{Tt}$

From Ex. 8.6a : $(1-B^d) Y_{St} = (1-B) Z_{St}$

$$(1-B)^2 Y_{Tt} = (1+\theta_1 B + \theta_2 B^2) Z_{Tt} \Rightarrow Y_{Tt} = (1-B)^{-2}(1+\theta_1 B + \theta_2 B^2) Z_{Tt}$$

$$(1-B^d) Y_{St} = (1-B) Z_{St} \Rightarrow Y_{St} = (1-B^d)^{-1}(1-B) Z_{St}$$

$$\Downarrow$$

$$Y_t = Y_{Tt} + Y_{St} = (1-B)^{-2}(1+\theta_1 B + \theta_2 B^2) Z_{Tt} + (1-B^d)^{-1}(1-B) Z_{St}$$

$$\Downarrow$$

$$(1-B)(1-B^d) Y = \frac{1-B^d}{1-B}(1+\theta_1 B + \theta_2 B^2) Z_{Tt} + (1-B)^2 Z_{St}$$

$$\Downarrow$$

$$(1-B)(1-B^d) Y = (1 + B + B^2 + \cdots + B^{d-1})(1+\theta_1 B + \theta_2 B^2) Z_{Tt} + (1-B)^2 Z_{St}$$

The right hand side is d+1 correlated, and can therefore be rewritten as an equivalent MA(d+1) representation, as should be shown.

Exercise 8.9

Combining two State-Space models

Model 1: $\begin{cases} \mathbf{X}_{t+1}^1 = \mathbf{F}^1 \mathbf{X}_t^1 + \mathbf{V}_t^1 \\ \mathbf{Y}_t^1 = \mathbf{G}^1 \mathbf{X}_t^1 + \mathbf{W}_t^1 \end{cases}$

Model 2: $\begin{cases} \mathbf{X}_{t+1}^2 = \mathbf{F}^2 \mathbf{X}_t^2 + \mathbf{V}_t^2 \\ \mathbf{Y}_t^2 = \mathbf{G}^2 \mathbf{X}_t^2 + \mathbf{W}_t^2 \end{cases}$

$$\begin{pmatrix} \mathbf{V}_t^1 \\ \mathbf{W}_t^1 \\ \mathbf{V}_t^2 \\ \mathbf{W}_t^2 \end{pmatrix} \text{WN}(0, \mathbf{\Sigma})$$

Superscripts denotes the respective models.

Define the joint observ. vector: $\mathbf{Y}_t = \begin{pmatrix} \mathbf{Y}_t^1 \\ \mathbf{Y}_t^2 \end{pmatrix}$

Observ. equation: $\mathbf{Y}_t = \begin{pmatrix} \mathbf{Y}_t^1 \\ \mathbf{Y}_t^2 \end{pmatrix} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \end{pmatrix} \begin{pmatrix} \mathbf{X}_t^1 \\ \mathbf{X}_t^2 \end{pmatrix} + \begin{pmatrix} \mathbf{V}_t^1 \\ \mathbf{V}_t^2 \end{pmatrix} = \mathbf{G}\mathbf{X}_t + \mathbf{V}_t$

and the joint state vector: $\mathbf{X}_t = \begin{pmatrix} \mathbf{X}_t^1 \\ \mathbf{X}_t^2 \end{pmatrix}$

State equation: $\mathbf{X}_{t+1} = \begin{pmatrix} \mathbf{X}_{t+1}^1 \\ \mathbf{X}_{t+1}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \mathbf{X}_t^1 \\ \mathbf{X}_t^2 \end{pmatrix} + \begin{pmatrix} \mathbf{W}_t^1 \\ \mathbf{W}_t^2 \end{pmatrix} = \mathbf{F}\mathbf{X}_t + \mathbf{W}_t$

$\mathbf{V}_t$ and $\mathbf{W}_t$ indep. **iff** $\mathbf{V}_t^1$ and $\mathbf{W}_t^2$ as well as $\mathbf{V}_t^2$ and $\mathbf{W}_t^1$ are indep.
($\mathbf{V}_t^1$ and $\mathbf{W}_t^1$ and $\mathbf{V}_t^2$ and $\mathbf{W}_t^2$ are indep. since Model 1 and 2 are S - S repr.)

If so, $\mathbf{W}_t$ and $\mathbf{V}_t$ are independent WN and we have a state-space representation for $\mathbf{Y}_t$

Exercise 8.9 cont'd.

Corollary: S-S representation for the sum of two univariate series:

$$\text{Model } 1: \begin{cases} \mathbf{X}^1_{\mathbf{t+1}} = \mathbf{F}^1 \mathbf{X}^1_{\mathbf{t}} + \mathbf{V}^1_{\mathbf{t}} \\ Y^1_t = \mathbf{G}^1 \mathbf{X}^1_{\mathbf{t}} + W^1_t \end{cases} \qquad \text{Model } 2: \begin{cases} \mathbf{X}^2_{\mathbf{t+1}} = \mathbf{F}^2 \mathbf{X}^2_{\mathbf{t}} + \mathbf{V}^2_{\mathbf{t}} \\ Y^2_t = \mathbf{G}^2 \mathbf{X}^2_{\mathbf{t}} + W^2_t \end{cases} \qquad \begin{pmatrix} \mathbf{V}^1_{\mathbf{t}} \\ \hdashline W^1_t \\ \hdashline \mathbf{V}^2_{\mathbf{t}} \\ \hdashline W^2_t \end{pmatrix} \text{WN}(0, \mathbf{\Sigma})$$

Observ. equation:

$$Y_t = Y^1_t + Y^2_t = (1 \;\; 1) \begin{pmatrix} Y^1_t \\ Y^2_t \end{pmatrix}$$

$$= (1 \;\; 1) \left(\begin{array}{c:c} \mathbf{G}_1 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{G}_2 \end{array} \right) \begin{pmatrix} \mathbf{X}^1_{\mathbf{t}} \\ \hdashline \mathbf{X}^2_{\mathbf{t}} \end{pmatrix} + (1 \;\; 1) \begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix} = \left(\mathbf{G}_1 \;\vdots\; \mathbf{G}_2 \right) \begin{pmatrix} \mathbf{X}^1_{\mathbf{t}} \\ \hdashline \mathbf{X}^2_{\mathbf{t}} \end{pmatrix} + W^1_t + W^2_t$$

$$= \mathbf{G}\mathbf{X}_{\mathbf{t}} + V_t \,, \text{ where } \; W_t = W^1_t + W^2_t$$

with the same state equation as before.