Non-linear programing and multiple constraints

ECON 3120/4120 Autumn 2013, UiO

This note, on non-linear maximization problems, has two purposes.

1. To describe 2 different methods of finding admissible candidates that satisfy the Kuhn-Tucker conditions.

2. To help you familiarize with the complementary slackness conditions.

Consider this maximization problem, with the admissible set sketched in Figure 1.

$$\max\ f(x,y) \quad s.t. \quad \begin{cases} g_1(x,y) \leq c_1 \\ g_2(x,y) \leq c_2 \\ g_3(x,y) \leq c_3 \end{cases}$$

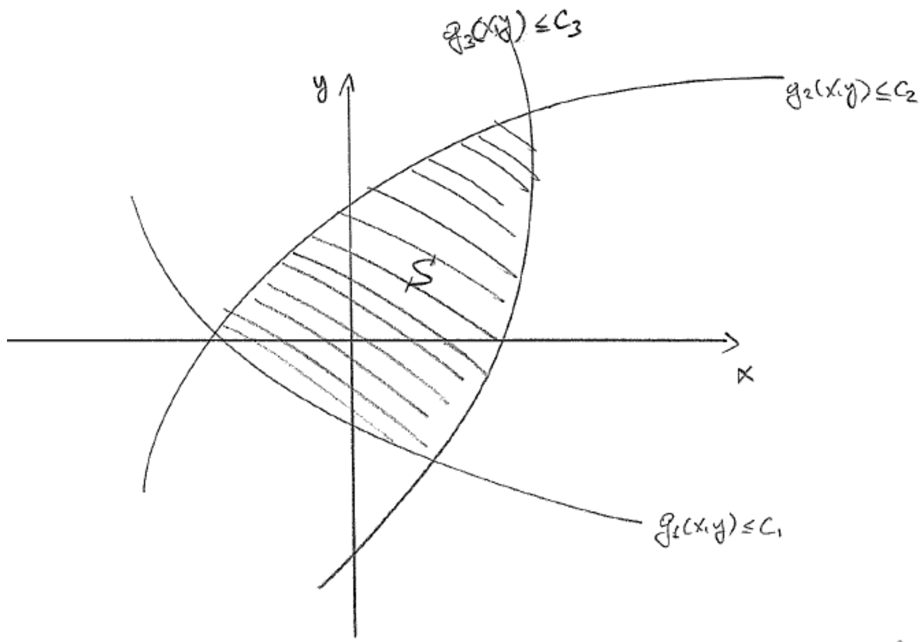

Figure 1: Admissible set

Suppose that you are asked to find all *admissible* points that satisfy the Kuhn-Tucker conditions. (*Non-admissible points are not interesting!*)

$\lambda_1,\ \lambda_2,\ \lambda_3$ are the multipliers associated with constraints 1, 2 and 3 respectively and the Lagrange function is:

$$L(x, y) = f(x, y) - \lambda_1[g_1(x, y) - c_1] - \lambda_2[g_2(x, y) - c_2] - \lambda_3[g_3(x, y) - c_3]$$

Before you continue reading, write down the Kuhn-Tucker conditions and the admissibility conditions.

1 Active vs inactive constraints

Any admissible point falls into one of the following four categories. A method for finding the admissible candidates that satisfy the Kuhn-Tucker conditions, is to consider each of the cases separately.

- Points where no constraints are active
- Points where only 1 constraint is active
- Points where 2 constraints are active
- Points where 3 constraints are active

See 14.9 in EMEA or 8.8 in MAII for an example with two constraints.

1.1 Points where no constraints are active

These are the interior points. By the complementary slackness conditions, such points have multipliers equal to zero (i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 0$).

This means that any interior points that satisfy the Kuhn-Tucker conditions must be stationary points for $f(x, y)$. (*Check the Kuhn-Tucker conditions to convince yourself of this.*) And remember that stationary points, by definition, satisfy the following system of equations:

$$\begin{cases} f'_x(x, y) = 0 \\ f'_y(x, y) = 0 \end{cases}$$

Hence, we can say that interior, stationary points satisfy the Kuhn-Tucker conditions with all multipliers equal to zero. (In fact, this holds for all admissible stationary points, whether they are interior points or boundary points.)

Claim: All *unconstrained* maximum, minimum and saddle points of $f(x, y)$ satisfy the Kuhn-Tucker conditions. (*Q1: Why?*)

1.2 Points where only 1 constraint is active.

Here we have 3 separate sub-cases to consider. In each of these sub-cases, there will be at least 2 multipliers equal to zero.

For example, if only the first constraint is active we get:

$$\begin{cases} g_1(x,y) = c_1 \\ g_2(x,y) < c_2 \\ g_3(x,y) < c_3 \end{cases}$$

This means that $\lambda_2 = \lambda_3 = 0$. (*Q2: Why?*)

If there are any admissible points here, with $\lambda_1 \geq 0$ then these are candidates that satisfy the Kuhn-tucker conditions. This means that you need to find the value of the first multiplier and make sure that it is non-negative before you can claim that you have found a candidate.

Note: You may find $\lambda_1 = 0$. In that case you have found a stationary point, which is not in the interior of the admissible set S.

1.3 Points where only 2 constraints are active

This gives 3 cases, namely the 3 intersection points. It remains to check whether they have non-negative multipliers.

For example, let's consider the case with only constraints 2 and 3 active.

$$\begin{cases} g_1(x,y) = c_1 \\ g_2(x,y) = c_2 \\ g_3(x,y) < c_3 \end{cases}$$

Then the multiplier associated with constraint 3 is zero (i.e. $\lambda_3 = 0$). (*Q3: Why?*) You need to check whether the point has $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

1.4 Points where 3 constraints are active

According to the graph, there are no such points.

2 An alternative

Another method of solving the problem is the following.

Search for admissible candidates in each of these cases:

- All multipliers equal to zero ($\lambda_1 = \lambda_2 = \lambda_3 = 0$)
- Only 2 multipliers equal to zero (3 sub-cases)
- Only 1 multiplier equal to zero (3 sub-cases)
- All multipliers non-zero ($\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0$)

Using the complementary slackness conditions, you can determine which constraints are active in each of the possible combinations (there is a total of 8).

Note that there is not a one-to-one correspondence between the cases in this method and the cases in the previous method. For example, the first bullet points in both methods consider stationary points. But only one of them includes stationary points on the boundary of the admissible set. (*Q4: Is that the first bullet point in method 1 or method 2?*)

3 What should you use?

Which of these methods you use is completely up to you.

Also, depending on the problem at hand, there may be other ways of solving, which may be more time-efficient. In Example 5, on the lecture of 11.09.2013, I used a somewhat different method than those above. I approached the problem by considering admissible stationary points first. This is equivalent to $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (i.e. the first bullet point in method 2). Then I deviated away from method 2, by considering the boundaries of the admissible set.

However, the advantage of the 2 methods described above is that they give a systematic approach to searching for candidates.

Example 5 from lecture on 11.09.2013.

Find all the (admissible) points that satisfy the Kuhn-Tucker conditions in the following maximization problem.

$$\max \frac{1}{2}(x^2 + y^2) \quad s.t. \quad \begin{cases} y \leq 1 - a(x-1) \\ x \geq 0 \\ y \geq 0 \end{cases} , \qquad for\ each\ a > 0$$

Find the points using one of the systematic methods, described in Parts 1 and 2 above.