UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Term paper in: ECON3120/4120 - Mathematics 2: Calculus and Linear Algebra

Handed out: 28.09.2016

To be delivered by: 12.10.2016 – within 3pm

Place of delivery: Fronter

Further instructions:

- This term paper is **compulsory**. Candidates who have passed the compulsory term paper in a previous semester, do not have the right to hand in the term paper again. This is so, even if the candidate did not pass the exam.

- You must deliver the term paper within the given deadline. Term papers delivered after the deadline, **will not be corrected**.*
- All term papers must be delivered in Fronter. Do not deliver your term paper to the course teacher or send it by e-mail.
- **Note:** The students can feel free to discuss with each other how to solve the problems, but each student is supposed to formulate her/his own answers. Only single-authored papers are accepted, and papers that for all practical purposes are identical will not be approved.

- Information about citing and referring to sources:
  http://www.uio.no/english/studies/admin/examinations/sources-citations/

- Information about consequences of cheating:
  http://www.uio.no/english/studies/admin/examinations/cheating/index.html

- If the term paper is not accepted, you will be given a new attempt (this will not apply if you deliver a blank page). If you still do not succeed, you will not be permitted to take the exam in this course. You will then be withdrawn from the exam.

*) If a student believes that she or he has a good cause not to meet the deadline (e.g. illness) she or he must seek a formal extension from the Department of Economics. Normally extension will only be granted when there is a good reason backed by supporting evidence (e.g. medical certificate).

ECON3120/4120 Mathematics 2

Compulsory term paper, autumn term 2016.

There are 3 pages of problems to be solved, not counting this page.

Justifying answers:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Minimum requirements to pass this assignment:

- You will pass if *each of problems 1, 2, and 3* is scored as good enough to pass, if it were judged to be one exam stand-alone.
  (The commonly applied pass mark in mathematics is forty percent, and this course by default uses uniform weighting over letter-enumerated items.)
- If you fail one problem despite a decent attempt at it, we may still let you pass upon judging the overall quality of the paper.

The paper does not count towards your final course grade!

- Passing the term paper is required in order to sit in on the exam (see the Department's rules). Other than that, it does not in any way count towards your grade, and the exam grading committee will not see your term paper.

Problem 1

(a) Let $h(x) = x^e \cdot [e^{sx} + e^{s^3 x}]$ and $g(x) = \frac{\ln(1+h(x))}{h(x)}$. Consider for each $s \neq 0$ the limits

$$\lim_{x \to 0^+} g(x) \qquad \text{and} \qquad \lim_{x \to +\infty} g(x)$$

For each $s \neq 0$: what does l'Hôpital's rule tell you about each of these limits?

(b) For s in a certain range, the function $h(x)$ of part (a) will have a global maximum point $x^* > 0$. The maximum value $V = h(x^*)$ depends on s. Find an expression for $V'(s)$.

(c) Let $u(x) = x + x^x$ and $v(x) = \log_x(1 + u(x))$. Find $v'(x)$.
(*Hint:* $x^{v(x)} = 1 + u(x)$.)

(d) An account accumulates interest as $e^{\rho t}$ continuously compounded, where t is time measured in years.

- Suppose first that $\rho = 0.025$. Find p so that the annual interest rate is $p\,\%$.
- Suppose instead that the annual interest rate is $e\,\%$ (where e is the constant ≈ 2.71828). Convert this to a continuously compounded ρ.

(e) Let f be a twice continuously differentiable function defined on $(0, \infty)$, and assume f is convex and that $\lim_{x \to 0^+} f(x) < 0$. Use proof by contradiction to establish that f cannot have two (nor more) zeroes.

Problem 2 Let $f(x,y) = e^{x^3y^2} - xy^2 - 1$.

(a) f has the following properties: (I): $f(x,y) = 0$ when $xy = 0$; (II): $f'_x(x,0) = f'_y(x,0) = 0$ for every x; (III): $f'_x(0,y) < 0$ for all $y \neq 0$.

i) Use the properties (I)–(III) to classify the stationary point $(0,0)$ *without* invoking the second-derivative test

ii) Can the second-derivative test classify the stationary point $(0,0)$?

Let from now on $t > 0$ be a constant, and consider the problem

$$\max\ f(x,y) \qquad \text{subject to} \qquad (tx+1)y \leq 1, \qquad x \geq 0 \qquad \text{and} \quad y \geq 0 \tag{P}$$

(b) i) State the Kuhn–Tucker conditions associated with problem (P).

ii) Do we know already, without further calculations, that there will be at least one point which satisfies these conditions?

(c) Consider the points such that $x = 0$ and $0 < y < 1$. Which of these points – if any – will satisfy the Kuhn–Tucker conditions?

(d) It can be shown – but you are not asked to do so – that if the Kuhn–Tucker conditions hold with $x > 0$ and $(tx+1)y = 1$, then

$$x^2 e^{x^3/(tx+1)^2} = \frac{tx+3}{tx+2}$$

Show that there exists a positive x satisfying this equation.

Problem 3 Let r, s, t, u, v be real constants with $s > 0, t > 0, u > 0$. Define the matrices

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{pmatrix} s & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

and denote by $\mathbf{I}$ the identity matrix and by $\mathbf{0}$ the null matrix of order 5×5.

(a) Calculate $\mathbf{U}^2$ and $(r\mathbf{L} + \mathbf{D} + u\mathbf{U})\mathbf{L}$.

(b) Which of the determinants $|\mathbf{L}|$, $|\mathbf{L}' + \mathbf{L}|$, $|\mathbf{D}|$ and $|(2\mathbf{DULL}')^4(\mathbf{I} + \mathbf{L}' + \mathbf{U})^{2016}|$ will be zero? (The prime ($'$) denotes transpose; recall that s, t and u are all > 0.)

(c) Put $s = t = 1$. In this part, you shall solve for the inverse of $(\mathbf{D} + u\mathbf{U})$ or show that it does not exist, for every $u > 0$. That is, you shall solve the equation system $\mathbf{AX} = \mathbf{I}$, where $\mathbf{A} = \mathbf{D} + u\mathbf{U}$. Note that the unknown $\mathbf{X}$ is 5×5.

- For full score: write down the augmented coefficient matrix $(\mathbf{A} \vdots \mathbf{I})$ and perform Gaussian elimination on this until you have found the solution or shown that no solution exists.
- For up to 2/3 score (usually corresponding to a near-middle-of-the-road "C"): solve the three equation systems $\mathbf{Ax} = \big(1,\ 0,\ 0,\ 0,\ 0\big)'$, $\mathbf{Ay} = \big(0,\ 1,\ 0,\ 0,\ 0\big)'$ and $\mathbf{Az} = \big(0,\ 0,\ 1,\ 0,\ 0\big)'$ and explain then where/how $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ enters the inverse and how you would solve the rest.

(d) *(This part involves concepts outside Mathematics 2 but you should be very well able to solve it using only Mathematics 2 curriculum. Note also that the requirement to pass every problem is applied to problems 1, 2 ... and not to each letter-enumerated item.)*

In Mathematics 3 – as well as in the first-semester bachelor course MAT1001 which is counted as eighty percent overlapping with ECON2200 – one introduces the concepts of *eigenvectors and eigenvalues* of square matrices: $\mathbf{v} \neq \mathbf{0}$ is an eigenvector of $\mathbf{M}$ if there exists some number λ such that $\mathbf{Mv} = \lambda\mathbf{v}$. This λ is then called the eigenvalue.

- If $\mathbf{v}$ is an eigenvector of $\mathbf{M}$ with corresponding eigenvalue λ, how many solutions will then the equation $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ have? Zero, one or infinitely many? (Note that $\mathbf{x}$ is the unknown, and λ is the fixed number.)
- Which of the above matrices $\mathbf{L}$, $\mathbf{U}$ and $\mathbf{D}$ has/have $\mathbf{w} = (0, 0, 1, 0, 0)'$ as eigenvector? (Hint: What do you get when you calculate $\mathbf{Lw}$?)
