Department of Economics
February 2006
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ECON3120/4120 Mathematics 2, spring 2006

Problem solutions for seminar no. 4, 20–24 February 2006

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA 9.6.2 (= MA I, 10.7.2)

(b) With $u = g(x) = x^3 + 2$ we get $du = g'(x)\,dx = 3x^2\,dx$ and

$$\int x^2 e^{x^3+2}\,dx = \int e^{g(x)}\frac{1}{3}g'(x)\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{e^{x^3+2}}{3} + C\,.$$

(c) As a first attempt we could use the substitution $u = g(x) = x+2$, which gives $du = dx$ og

$$\int \frac{\ln(x+2)}{2x+4}\,dx = \int \frac{\ln u}{2u}\,du\,.$$

This does not look very much simpler than the original integral, but if we notice that $\dfrac{\ln u}{u} = \ln u \cdot \dfrac{1}{u} = \ln u \cdot \dfrac{d}{du}\ln u$, then we can see that $v = \ln u$ yields $dv = \dfrac{1}{u}\,du$ and

$$\int \frac{\ln u}{2u}\,du = \int \frac{1}{2}v\,dv = \frac{1}{4}v^2 + C = \frac{1}{4}(\ln u)^2 + C = \frac{1}{4}(\ln(x+2))^2 + C\,.$$

With a little experience we would have noticed straight away that

$$\frac{\ln(x+2)}{2x+4} = \frac{\ln(x+2)}{2(x+2)} = \frac{1}{2}\ln(x+2)\,\frac{d}{dx}\ln(x+2)\,,$$

and this immediately suggests the substitution $v = \ln(x+2)$.

EMEA 9.6.4 (= MA I, 10.7.4)

We want to solve the equation

$$\int_3^x \frac{2t-2}{t^2-2t}\,dt = \ln\Bigl(\frac{2}{3}x - 1\Bigr). \qquad (*)$$

For the right-hand side to make sense, we must have $\frac{2}{3}x > 1$, that is, $x > \frac{3}{2}$. We substitute a new variable in order to calculate the integral on the left-hand side:

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With $u = t^2 - 2t$, we get $du = (2t - 2)\,dt$. For $t = 3$ and $t = x$, we get $u = 3$ and $u = x^2 - 2x$, respectively, so the integral becomes

$$\int_3^x \frac{2t-2}{t^2-2t}\,dt = \int_3^{x^2-2x} \frac{du}{u} = \Big|_3^{x^2-2x} \ln u = \ln\Big(\frac{x^2-2x}{3}\Big).$$

Equation $(*)$ then yields

$$\frac{x^2-2x}{3} = \frac{2}{3}x - 1 \iff x^2 - 2x = 2x - 3 \iff x^2 - 4x + 3 = 0. \qquad (**)$$

The roots of equation $(**)$ are $x_1 = 3$ and $x_2 = 1$. Here x_2 is unusable as a solution of the original equation, because we have seen that we must have $x > 3/2$. (Also, the integral on the left would then become

$$\int_3^1 \frac{2t-2}{t^2-2t}\,dt = \int_3^1 \Big(\frac{1}{t} + \frac{1}{t-2}\Big)\,dt,$$

which does not converge.)

Thus we are left with the solution $x = 3$, and it is easy to check that this is indeed a solution of $(*)$. (Both sides of the equation become equal to 0.)

EMEA 9.7.3 (= MA I, 10.9.3)

(a) Let us first use integration by parts to calculate the indefinite integral, using the fact that $\lambda e^{\lambda x} = (d/dx)(-e^{-\lambda x})$:

$$\begin{aligned}\int x\lambda e^{-\lambda x}\,dx &= x(-e^{-\lambda x}) - \int 1 \cdot e^{-\lambda x}\,dx = -xe^{-\lambda x} + \int e^{-\lambda x}\,dx \\ &= -xe^{-\lambda x} - \frac{1}{\lambda}\,e^{-\lambda x} + C\end{aligned}$$

This yields

$$\begin{aligned}\int_0^b x\lambda e^{-\lambda x}\,dx &= \Big|_0^b -xe^{-\lambda x} - \frac{1}{\lambda}\,e^{-\lambda x} \\ &= \Big(-be^{-\lambda b} - \frac{1}{\lambda}\,e^{-\lambda b}\Big) - \Big(-0 - \frac{1}{\lambda}e^0\Big) = \frac{1}{\lambda} - be^{-\lambda b} - \frac{1}{\lambda}\,e^{-\lambda b},\end{aligned}$$

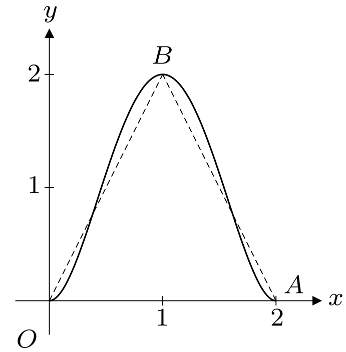
and so

$$\int_0^\infty x\lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} \int_0^b x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}\,.$$

(We are assuming here that λ is a positive constant, as in Example 1 on page 335 (page 361 in MA I). We have also used that $be^{-\lambda b} \to 0$ as $b \to \infty$. This follows easily from using l'Hôpital's rule for "∞/∞"-expressions on $b/e^{\lambda b}$, or from the general result in formula (4) on page 264 (formula (4) on page 224 in MA I), with b instead of x.)

Exam problem 53

(a) $$\int_0^2 2x^2(2-x)^2\,dx = \int_0^2 (8x^2 - 8x^3 + 2x^4)\,dx = \Big|_0^2 \Big(\frac{8}{3}x^3 - 2x^4 + \frac{2}{5}x^5\Big) = \frac{8}{3}\cdot 8 - 2\cdot 16 + \frac{2}{5}\cdot 32 = \frac{32}{15} = 2\frac{2}{15}$$



We can see from the figure that the area between the graph and the x-axis over the interval $[0, 2]$ must be approximately equal to the area of the triangle OAB, where the point O, A, and B are $(0, f(0)) = (0, 0)$, $(2, f(2)) = (2, 0)$, and $(1, f(1)) = (1, 2)$, respectively. The area of this triangle is exactly 2.

(b) Here we shall *not* try to find the function x. (That would require knowledge of trigonometric and inverse trigonometric functions.) Instead we shall try to see if there is some other way to find the information we need in order to show that $t = 0$ is a minimum point for x. It turns out to be fairly easy in this case. In fact, $\dot{x}(t) < 0$ for $t < 0$ and $\dot{x}(t) > 0$ for $t > 0$. Hence, x is strictly decreasing in $(-\infty, 0]$ and strictly increasing in $[0, \infty)$, so $t = 0$ must be a global minimum point for $x = x(t)$. Note that this gives $x(t) \geq x(0) = 0$ for all t.

Furthermore,

$$\ddot{x} = \frac{d}{dt}\big((1 + x^2)t\big) = 2x\dot{x}t + (1 + x^2) = 2x(1 + x^2)t^2 + (1 + x^2).$$

Since $x(t) \geq 0$ for all t, we have $\ddot{x}(t) \geq 1 > 0$ for all t. It follows that x is (strictly) convex.

(c) Taking elasticities with respect to x gives

$$\begin{aligned} \mathrm{El}_x\, x^a + \mathrm{El}_x\, y^b &= \mathrm{El}_x\, A + \mathrm{El}_x\, e^{x/y^2}, \\ a + b\,\mathrm{El}_x\, y &= 0 + \mathrm{El}_u\, e^u\, \mathrm{El}_x\, u \qquad \text{(with } u = x/y^2\text{)} \\ &= u(\mathrm{El}_x\, x - \mathrm{El}_x\, y^2) \\ &= \frac{x}{y^2}(1 - 2\,\mathrm{El}_x\, y) \end{aligned}$$

and consequently,

$$\mathrm{El}_x\, y = \frac{x - ay^2}{2x + by^2}\,.$$

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Exam problem 77

(i) We first calculate the indefinite integral. Integration by parts gives

$$\int x(2+x)^{1/3}\,dx = x\frac{3}{4}(2+x)^{4/3} - \frac{3}{4}\int (2+x)^{4/3}\,dx$$
$$= x\frac{3}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3} + C$$

The definite integral is then

$$\int_{-1}^{6} x(2+x)^{1/3}\,dx = \Big|_{-1}^{6}\left(\frac{3x}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3}\right)$$
$$= \frac{9}{2}8^{4/3} - \frac{9}{28}8^{7/3} - \left(-\frac{3}{4} - \frac{9}{28}\right) = \frac{447}{14} \approx 31.92,$$

where we have used that $8^{1/3} = \sqrt[3]{8} = 2$.

Alternatively, we can use substitution and calculate as follows: Introduce $u = (2+x)^{1/3}$ as a new variable. That gives $x = u^3 - 2$, $dx = 3u^2\,du$, and

$$\int x(2+x)^{1/3}\,dx = \int (u^3-2)u3u^2\,du = \int (3u^6 - 6u^3)\,du$$
$$= \frac{3}{7}u^7 - \frac{6}{4}u^4 + C = \frac{3}{7}(2+x)^{7/3} - \frac{3}{2}(2+x)^{4/3} + C.$$

(This is indeed equal to the indefinite integral we found above, although it does not look that way at first glance.)

We the calculate the definite integral as before. However, we can also use formula (2) on page 333 in EMEA (page 355 in MA I). That will give us

$$\int_{-1}^{6} x(2+x)^{1/3} = \int_{1}^{2} (3u^6 - 6u^3)\,du = \Big|_{1}^{2}\left(\frac{3}{7}u^7 - \frac{3}{2}u^4\right)$$

etc.

(ii) Here we use the substitution $z = \sqrt[3]{x} = x^{1/3}$, which gives $x = z^3$ and $dx = 3z^2\,dz$. The integral then becomes

$$\int e^{\sqrt[3]{x}}\,dx = \int e^z 3z^2\,dz = 3\int z^2 e^z\,dz.$$

In order to find the last integral, we use integration by parts twice:

$$\int z^2e^z\,dz = z^2e^z - \int 2ze^z\,dz = z^2e^z - \left(2ze^z - \int 2e^z\,dz\right)$$
$$= z^2e^z - 2ze^z + \int 2e^z\,dz = z^2e^z - 2ze^z + 2e^z + C.$$

Then

$$\int e^{\sqrt[3]{x}}\,dx = 3(z^2e^z - 2ze^z + 2e^z + C) = (3x^{2/3} - 6x^{1/3} + 6)e^{\sqrt[3]{x}} + C_1,$$

where $C_1 = 3C$.

Exam problem 88

(a) (i) Integration by parts yields

$$\int 3xe^{-x/2}\,dx = 3x(-2e^{-x/2}) + \int 6e^{-x/2}\,dx = -6xe^{-x/2} - 12e^{-x/2} + C.$$

(ii) The substitution $u = 9 + \sqrt{x}$ gives $x = (u-9)^2$ and $dx = 2(u-9)\,du$. As x runs from 0 to 25, u runs from 9 to 14, and so we have

$$\int_0^{25} \frac{1}{9+\sqrt{x}}\,dx = \int_9^{14} \frac{2(u-9)}{u}\,du = \int_9^{14}\Big(2 - \frac{18}{u}\Big)\,du = 10 - 18\ln\frac{14}{9}\,.$$

(iii) We introduce $u = \sqrt{t+2}$ as a new variable and get $t = u^2 - 2$ and $dt = 2u\,du$. This yields

$$\int_2^7 t\sqrt{t+2}\,dt = \int_2^3 (u^2-2)u\cdot 2u\,du = \int_2^3 (2u^4 - 4u^2)\,du$$
$$= \Big|_2^3 \Big(\frac{2}{5}u^5 - \frac{4}{3}u^3\Big) = \Big(\frac{486}{5} - \frac{108}{3} - \frac{64}{5} + \frac{32}{3}\Big) = \frac{886}{15}\,.$$

(b) This is a separable differential equation. It has one constant solution, $x \equiv 0$. The other solutions are given by

$$(*) \qquad \int \frac{dx}{x} = \int \frac{4(a-t)}{(2t-a)^2}\,dt.$$

The integral on the left is $\displaystyle\int \frac{dx}{x} = \ln|x| + C_1$. In order to calculate the integral on the right, we introduce a new variable. With $u = 2t - a$ we get $t = (u+a)/2$ and $dt = \frac{1}{2}\,du$. This yields $a - t = (a-u)/2$ and

$$\int \frac{4(a-t)}{(2t-a)^2}\,dt = \int \frac{2(a-u)}{u^2}\,\frac{1}{2}\,du = \int \frac{a-u}{u^2}\,du = \int\Big(\frac{a}{u^2} - \frac{1}{u}\Big)du$$
$$= -\frac{a}{u} - \ln u + C_2 = -\frac{a}{2t-a} - \ln(2t-a) + C_2.$$

(u is positive, since $t > a/2$.) Hence,

$$\ln|x| = -\frac{a}{2t-a} - \ln(2t-a) + C \qquad (C = C_2 - C_1),$$

so

$$x = Be^{-a/(2t-a)} e^{-\ln(2t-a)} = \frac{B}{2t-a} e^{-a/(2t-a)} .$$

The constant B equals $\pm e^C$, where the sign is the same as the sign of x. For $B = 0$ we get the constant solution $x \equiv 0$.

Exam problem 111

This is a separable differential equation, and we get

$$\int \frac{3x^2}{(x^3+9)^{3/2}}\, dx = \int \ln t \, dt.$$

(Because we are looking for the solution through $(t, x) = (1, 3)$, we need not worry about the single constant solution, $x \equiv -\sqrt[3]{9}$.) With $u = x^3+9$, we get $du = 3x^3\, dx$ and

$$\int \frac{3x^2}{(x^3+9)^{3/2}}\, dx = \int \frac{du}{u^{3/2}} = \int u^{-3/2}\, du$$
$$= -2u^{-1/2} + C_1 = -\frac{2}{\sqrt{x^3+9}} + C_1 .$$

Integration by parts yields

$$\int \ln t = \int 1 \cdot \ln t = t \ln t - t + C_2,$$

cf. Problem 9.5.3 in EMEA (10.6.3 in MA I). Alternatively, we could use the substitution $v = \ln t$, which yields $t = e^v$, $dt = e^v\, dv$, and

$$\int \ln t\, dt = \int ve^v\, dv = ve^v - \int 1 \cdot e^v\, dv$$
$$= ve^v - e^v + C_2 = t \ln t - t + C_2 .$$

(But note that we still do not avoid integration by parts.) Hence,

$$-\frac{2}{\sqrt{x^3+9}} = t \ln t - t + C , \quad \text{where} \quad C = C_2 - C_1 .$$

The graph of the solution passes through $(t, x) = (1, 3)$ if and only if

$$-\frac{2}{\sqrt{x^3+9}} = -1 + C ,$$

i.e. $C = 1 - \frac{2}{6} = \frac{2}{3}$. This yields

$$-\frac{2}{\sqrt{x^3+9}} = t \ln t - t + \frac{2}{3} .$$

Let $\varphi(t) = t\ln t - t + 2/3$. Then,

$$-\frac{2}{\sqrt{x^3+9}} = \varphi(t) \Longrightarrow \sqrt{x^3+9} = -\frac{2}{\varphi(t)} \Longrightarrow x^3+9 = \frac{4}{\varphi(t)^2}$$
$$\Longrightarrow x = \Big(\frac{4}{\varphi(t)^2} - 9\Big)^{1/3}.$$

Exam problem 137

(a) $$\int_4^9 \frac{(\sqrt{x}-1)^2}{x}\,dx = \int_4^9 \frac{x-2\sqrt{x}+1}{x}\,dx = \int_4^9 (1-2x^{-1/2}+\frac{1}{x})\,dx$$
$$= \Big|_4^9 (x-4\sqrt{x}+\ln x) = 9-4\cdot 3+\ln 9-4+4\cdot 2-\ln 4 = 1+\ln\frac{9}{4}\,.$$

(b) We introduce the new variable $u = 1+\sqrt{x}$. Then $u-1=\sqrt{x}$, $(u-1)^2 = x$, and $2(u-1)\,du = dx$. When $x=0$, then $u=1$, and when $x=1$, then $u=2$. Hence

$$\begin{aligned} I &= \int_1^2 (\ln u)2(u-1)\,du = 2\int_1^2 u\ln u\,du - 2\int_1^2 \ln u\,du \\ &= 2\Big(\Big|_1^2 \frac{1}{2}u^2\ln u - \int_1^2 \frac{1}{2}u^2\frac{1}{u}\,du\Big) - 2\Big|_1^2 (u\ln u - u) \\ &= \Big|_1^2 (u^2\ln u - \frac{1}{2}u^2) - 2\Big|_1^2 (u\ln u - u) \\ &= 4\ln 2 - 2 - 0 + \frac{1}{2} - 2(2\ln 2 - 2 + 1) = \frac{1}{2}\,. \end{aligned}$$

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