EXAM PROBLEMS

IN

ECON4140/4145 MATHEMATICS 3

DIFFERENTIAL EQUATIONS,

STATIC AND DYNAMIC OPTIMIZATION

Department of Economics, University of Oslo
2008

Preface

The problems in this collection are (mostly) taken from exams in mathematics for economists at the Department of Economics, University of Oslo. Answers to the problems are supplied at the back. A few minor errors in the 2004 printing have been corrected. We thank Li Cen for excellent help with this booklet.

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Oslo, July 2005,

Arne Strøm and Knut Sydsæter

For this new edition we have added two sections of problems on difference equations and dynamic programming.

Oslo, April 2008,

Arne Strøm and Knut Sydsæter

Contents

4145ex3 29.4.2008 621

1. Linear Algebra

Problem 1-01

(a) Let the matrix $\mathbf{A}$ be defined by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. Compute $\mathbf{A}^2$ and $\mathbf{A}^3$.

(b) Find the eigenvalues of $\mathbf{A}$ and corresponding eigenvectors.

(c) Let $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$. Compute $\mathbf{P}^{-1}$, and show that $\mathbf{A} = \mathbf{P}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{P}^{-1}$.

Problem 1-02

(a) Find the eigenvalues of the matrix $\mathbf{A}_a = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & -a \\ 2-a & 1 & 2 \end{pmatrix}, \quad a \leq 1$

(b) Find corresponding eigenvectors in the cases (i) $a = 1$ and (ii) $a = -3$.

Problem 1-03

Given three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^n$.

(a) Are $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$ and $\mathbf{a} - \mathbf{c}$ linearly independent?

(b) Let $\mathbf{d} = 4\mathbf{a} - \mathbf{b} - \mathbf{c}$. Is it possible to find numbers x, y and z such that

$$x(\mathbf{a} - \mathbf{b}) + y(\mathbf{b} - \mathbf{c}) + z(\mathbf{a} - \mathbf{c}) = \mathbf{d}\,?$$

Problem 1-04

Let $\mathbf{A}$ be the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and a set of corresponding eigenvectors of $\mathbf{A}$.

(b) Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, ... be a sequence of vectors given by

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t \quad \text{for } t = 0, 1, 2, \ldots$$

Show that $\mathbf{x}_0$ can be written as a linear combination of eigenvectors of $\mathbf{A}$. Use this to find $\mathbf{x}_t$ for $t = 1, 2, 3, \ldots$

Problem 1-05

Determine the eigenvalues of the matrix

$$\mathbf{A}_a = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & -a \\ 2-a & 1 & 2 \end{pmatrix}, \qquad a \leq 1$$

Find also the corresponding eigenvectors when $a = 1$ and when $a = -3$.

4145ex3 29.4.2008 621

Problem 1-06

(a) Find the rank of $\mathbf{D}_t = \begin{pmatrix} t & 0 & 0 & 1 \\ 0 & 2 & t & 3 \\ 1 & -2 & t & 0 \\ 2t & 1 & 0 & 3 \end{pmatrix}$ for all values of t.

(b) Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be $n \times n$ matrices where $\mathbf{A}$ and $\mathbf{C}$ are invertible. Solve the following matrix equation for $\mathbf{X}$:

$$\mathbf{CB} + \mathbf{CXA}^{-1} = \mathbf{A}^{-1}$$

Problem 1-07

(a) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where a, b, and c are different from 0. Find $\mathbf{A}^{-1}$.

(b) Given a 3×3 matrix $\mathbf{B}$ whose column vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are mutually orthogonal and different from the zero vector. Put $\mathbf{A} = \mathbf{B}'\mathbf{B}$ and show that $\mathbf{A}$ is a diagonal matrix.

(c) Find $\mathbf{B}^{-1}$ expressed in terms of $\mathbf{A} = \mathbf{B}'\mathbf{B}$ and $\mathbf{B}$.

(d) Prove that the columns of $\mathbf{P} = \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$ are mutually orthogonal. Find $\mathbf{P}^{-1}$ by using the results above.

Problem 1-08

Consider the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(a) Show that the characteristic polynomial of $\mathbf{A}$ can be written in the form $(4 - \lambda)(\lambda^2 + a\lambda + b)$ for suitable constants a and b. Find the eigenvalues of $\mathbf{A}$.

(b) Show that $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$, and $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$ are eigenvectors of $\mathbf{A}$.

Let $\mathbf{C}$ be the matrix with the three vectors from part (b) as columns.

(c) Show that $\mathbf{CC}' = \mathbf{I}_3$ (the identity matrix of order 3), and use this to find the inverse of $\mathbf{C}$. Compute $\mathbf{C}^{-1}\mathbf{AC}$. (This will be a diagonal matrix.)

(d) Let $\mathbf{D} = \text{diag}(d_1, d_2, d_3)$ be a diagonal matrix, and let $\mathbf{B} = \mathbf{CDC}^{-1}$. Show that $\mathbf{B}^2 = \mathbf{CD}^2\mathbf{C}^{-1}$, and that $\mathbf{B}^2 = \mathbf{A}$ for suitable values of d_1, d_2 and d_3.

Problem 1-09

(a) Find the rank of $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 2 & 0 \\ 0 & -1 & 2 & -3 \\ 1 & 0 & -2 & 0 \end{pmatrix}$.

(b) For what values of x, y and z are the three vectors $(x, 1, 0, 1)$, $(2, y, -1, 0)$ and $(0, 2, 2x, z)$ linearly independent?

Problem 1-10

Let the matrices $\mathbf{A}_k$ and $\mathbf{P}$ be given by

$$\mathbf{A}_k = \begin{pmatrix} 1 & k & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 1/\sqrt{10} & -3/\sqrt{35} & 3/\sqrt{14} \\ 0 & 5/\sqrt{35} & 2/\sqrt{14} \\ 3/\sqrt{10} & 1/\sqrt{35} & -1/\sqrt{14} \end{pmatrix}$$

(a) Determine the rank of $\mathbf{A}_k$ for all values of k.

(b) Find the characteristic equation of $\mathbf{A}_k$ and determine the values of k that make all the eigenvalues real.

(c) Show that the columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}_3$, and compute the matrix product $\mathbf{P}'\mathbf{A}_3\mathbf{P}$.

Problem 1-11

(a) Consider the equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{15}x_5 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{25}x_5 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{35}x_5 &= c_3 \end{aligned} \tag{$*$}$$

where the coefficient matrix has rank 3 and $x_1, \ldots, x_5$ are the unknowns. Does $(*)$ always have a solution? And if so, how many degrees of freedom are there?

(b) Add the equation $a_{41}x_1 + \cdots + a_{45}x_5 + a_{46}x_6 = c_4$ to system $(*)$, where x_6 is an additional unknown. Describe possible solutions, including the degrees of freedom, in the new system. (Explicit solutions are not required.)

Problem 1-12

(a) Consider the matrix $\mathbf{D}(s) = \begin{pmatrix} 1 & 2s & 1 & 1 \\ -2 & 1 & -2 & 3s \\ 1 & 1-s & -1 & 5 \\ -1 & 2 & s & -3 \end{pmatrix}$. Find a necessary and sufficient condition for $\mathbf{D}(s)$ to have rank 4. What is the rank if $s = 1$?

(b) Determine the number of degrees of freedom for the equation system

$$\begin{aligned} x + 2y + z + w &= 0 \\ -2x + y - 2z + 3w &= 0 \\ x \quad - z + 5w &= 0 \\ -x + 2y + z - 3w &= 0 \end{aligned}$$

Problem 1-13

(a) Let $\mathbf{A}$ be a symmetric $n \times n$ matrix with $|\mathbf{A}| \neq 0$, let $\mathbf{B}$ be a $1 \times n$ matrix, and let $\mathbf{X}$ be an $n \times 1$ matrix. Consider the expression

$$(\mathbf{X} + \tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}')'\mathbf{A}(\mathbf{X} + \tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}') - \tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}' \tag{$*$}$$

Expand and simplify.

(b) Suppose that $\mathbf{A}$ is symmetric and positive definite (i.e. $\mathbf{Y}'\mathbf{A}\mathbf{Y} > 0$ for all $n \times 1$ matrices $\mathbf{Y} \neq \mathbf{0}$). Using $(*)$, find the matrix $\mathbf{X}$ that minimizes the expression $\mathbf{X}'\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{X}$.

4145ex3 29.4.2008 621

Problem 1-14

Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

(a) Find the rank of $\mathbf{A}$, show that $(\mathbf{AA}')^{-1}$ exists, and find this inverse.

(b) Compute the matrix $\mathbf{C} = \mathbf{A}'(\mathbf{AA}')^{-1}$, and show that $\mathbf{ACb} = \mathbf{b}$ for every 2×1 matrix (2-dimensional column vector) $\mathbf{b}$.

(c) Use the results above to find a solution of the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 1 \end{aligned}$$

(d) Consider in general a linear system of equations

$$\mathbf{Ax} = \mathbf{b}, \quad \text{where } \mathbf{A} \text{ is an } m \times n \text{ matrix, } m \leq n \qquad (*)$$

It can be shown that if $r(\mathbf{A}) = m$, then $r(\mathbf{AA}') = m$. Why does this imply that $(\mathbf{AA}')^{-1}$ exists? Put $\mathbf{C} = \mathbf{A}'(\mathbf{AA}')^{-1}$, and show that if $\mathbf{v}$ is an arbitrary $m \times 1$ vector, then $\mathbf{ACv} = \mathbf{v}$. Use this to show that $\mathbf{x} = \mathbf{Cb}$ must be a solution of $(*)$.

Problem 1-15

Define the matrix $\mathbf{A}_a$ for all real numbers a by $\mathbf{A}_a = \begin{pmatrix} a & 0 & 1 \\ a & a & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Compute the rank of $\mathbf{A}_a$ for all values of a.

(b) Find all eigenvalues and eigenvectors of $\mathbf{A}_0$. (NB! Here $a = 0$.) Show that eigenvectors corresponding to different eigenvalues are mutually orthogonal.

(c) Discuss the rank of the matrix product $\mathbf{A}_a\mathbf{A}_b$ for all values of a and b.

Problem 1-16

Let $\mathbf{A} = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$.

(a) Compute the eigenvalues of $\mathbf{A}$. (*Hint:* You can use (without proof) the formula

$$\begin{vmatrix} a+b & a & \dots & a \\ a & a+b & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & a+b \end{vmatrix} = b^{n-1}(na+b)$$

where n is the order of the determinant.)

(b) One of the eigenvalues has multiplicity 3. Find three linearly independent eigenvectors associated with this eigenvalue. Find also an eigenvector associated with the fourth eigenvalue.

4145ex3 29.4.2008 621

Problem 1-17

Given the matrices $\mathbf{A} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of $\mathbf{E}$.

(b) Find numbers p and q such that $\mathbf{A} = p\mathbf{I}_3 + q\mathbf{E}$.

(c) Show that if $\mathbf{x}_0$ is an eigenvector of $\mathbf{E}$, then $\mathbf{x}_0$ is also an eigenvector of $\mathbf{A}$.

(d) Find the eigenvalues of $\mathbf{A}$.

Problem 1-18

Given the matrix $\mathbf{A} = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -2 \\ -1 & -1 & 3 \end{pmatrix}$.

(a) Show that $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$, and find the corresponding eigenvalues.

(b) Let $\mathbf{B} = \mathbf{A}\mathbf{A}$. Show that $\mathbf{B}\mathbf{x}_2 = \mathbf{x}_2$ and $\mathbf{B}\mathbf{x}_3 = \mathbf{x}_3$. Is $\mathbf{B}\mathbf{x}_1 = \mathbf{x}_1$?

(c) Let $\mathbf{C}$ be an arbitrary $n \times n$ matrix such that $\mathbf{C}^3 = \mathbf{C}^2 + \mathbf{C}$. Show that if λ is an eigenvalue of $\mathbf{C}$, then $\lambda^3 = \lambda^2 + \lambda$. Show that $\mathbf{C} + \mathbf{I}_n$ has an inverse.

Problem 1-19

(a) Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} -a^2b & 0 & ab \\ 0 & c & 0 \\ -ab & 0 & b \end{pmatrix}$.

(b) Let $\mathbf{H}$ be a 3×3 matrix with eigenvalues λ_1, λ_2 and λ_3, and let α be a number $\neq 0$. Show that $\alpha\lambda_1$, $\alpha\lambda_2$ and $\alpha\lambda_3$ are eigenvalues of the matrix $\mathbf{K} = \alpha\mathbf{H}$.

(c) Find the eigenvalues of $\mathbf{B} = \dfrac{1}{1-a^2} \begin{pmatrix} -4a^2 & 0 & 4a \\ 0 & 1-a^2 & 0 \\ -4a & 0 & 4 \end{pmatrix}$, $\quad a \neq \pm 1$.

(d) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{D}$, where $\mathbf{B}$ is the matrix in (c). Find then a matrix $\mathbf{C}$ such that $\mathbf{C}^2 = \mathbf{B}$. (*Hint:* Find first a diagonal matrix $\mathbf{E}$ such that $\mathbf{E}^2 = \mathbf{D}$. Make use of the formula $\mathbf{P}\mathbf{E}^2\mathbf{P}^{-1} = \mathbf{P}\mathbf{E}\mathbf{P}^{-1}\mathbf{P}\mathbf{E}\mathbf{P}^{-1}$ to find $\mathbf{C}$ expressed in terms of $\mathbf{E}$ and $\mathbf{P}$.)

2. Multivariable Calculus

Problem 2-01

Decide if the following functions are (strictly) concave/convex, quasiconcave/quasiconvex in their domains. (In (a) and (c) you should not differentiate.)

(a) (i) $f(x,y) = 10x^{0.5}y^{0.3}$, $\quad x \geq 0$, $\; y \geq 0$ $\quad$ (ii) $g(x,y) = 10x^{1/3}y^{5/3}$, $\quad x \geq 0$, $\quad y \geq 0$

(b) $F(x_1, x_2, x_3) = 3 - x_1^2 - x_1x_2 - 2x_2^2 + x_1x_3 - 5x_3^2$

(c) $G(x,y) = \sqrt{\ln(x+y-4)}$. (Where is G defined?)

(d) $H(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 - 4x_1x_3 + x_2^2 + 2x_3^2$

Problem 2-02

For which values of the constants a and b is $f(x, y) = e^{ax+by^2}$ defined for all x and y concave/convex?

Problem 2-03

Let f be defined for all x and y by $f(x, y) = 2x - y - e^x - e^{x+2y}$. Is f concave/convex?

Problem 2-04

Prove that $f(x, y) = e^{-x}\sqrt{1 + y^2}$ is strictly convex for $|y| < 1$.

Problem 2-05

Define the function f for all x, y by $f(x, y) = 3 \sin 2x + 3y - 3y^2 + 8$. Show that f is concave in the rectangle $R = \{ (x, y) : 0 < x < \pi/2,\ 0 < y < 1\}$.

Problem 2-06

Let D be the set of points (x, y) with $-1 < x < 1$ and $-1 < y < 1$, and let

$$f(x, y) = \frac{1}{12}(x - y)^4 - (x - y)^2 - (x + y)^2$$

(a) Show that f is neither convex nor concave in D.

(b) Find the subset of D at which f is concave.

Problem 2-07

Let $f(x, y) = (\ln x)^a (\ln y)^b$, defined for $x > 1,\ y > 1$. Assume that $a > 0,\ b > 0$ and $a + b < 1$. Compute the Hessian matrix $\mathbf{H}$ of f and show that f is strictly concave.

3. Static Optimization. Kuhn-Tucker Theory

Problem 3-01

Solve the problem $\max x^{1/2} y^{1/3}$ subject to $3x + 4y \leq 25$.

Problem 3-02

Consider the problem

$$\max\ f(x, y) = x^2 + y^2 \quad \text{subject to} \quad g(x, y) = 5x^2 + 6xy + 5y^2 \leq 1$$

(a) Write out the necessary Kuhn–Tucker conditions for a point (x, y) to solve the problem, and find all points that satisfy the conditions. Do any of those points solve the problem?

(b) The admissible set is the set of points inside an ellipse in the xy-plane. Explain why the problem can be interpreted as that of finding the point or points on the ellipse $g(x, y) = 1$ with the greatest distance from the origin.

(c) Determine the approximate changes in the maximum value of $f(x, y)$ if the constraint $g(x, y) \leq 1$ is replaced by $g(x, y) \leq 1.1$.

Problem 3-03

(a) Write down the Kuhn–Tucker conditions for the problem

$$\max\ x + xy \quad \text{subject to} \quad y + x^2 e^y \le 1$$

(b) Prove that $(0, -1)$ and $(1, 0)$ both satisfy the conditions and find the associated Lagrange multipliers.

Problem 3-04

Solve the following problem:

$$\text{minimize } f(x, y) = e^{x+y} + e^y + 2x + y \quad \text{s.t.} \quad x \ge -1,\ \ y \ge -1,\ \ x + y \ge 0$$

Problem 3-05

Consider the problem

$$\text{maximize } x^2 y e^{-x-y} \quad \text{s.t.} \quad x \ge 1,\ \ y \ge 1,\ \ x + y \ge 4$$

(a) Write down the Kuhn–Tucker conditions for the problem.

(b) Find all the solutions to these conditions.

(c) (Optional.) Have you in (b) found the solution to the problem?

Problem 3-06

(a) Solve the nonlinear programming problem

$$\text{maximize } \ f(x, y, z) = x + \ln(1 + z) \quad \text{subject to} \quad \begin{cases} x^2 + y^2 \le 1 \\ x + y + z \le 1 \end{cases}$$

(b) What is the approximate change in the optimal value of $f(x, y, z)$ if the second constraint is replaced by $x + y + z \le 1.02$?

Problem 3-07

Consider the problem

$$(*) \qquad \max\ (2x + y) \quad \text{s.t.} \quad \begin{cases} (x+1)^2 + y^2 \le 4 \\ x^2 + (y+1)^2 \le 4 \end{cases} \qquad x \ge 0,\ y \ge 0$$

(a) Let S be the set of all (x, y) which satisfy all the four constrains. Sketch S in the xy-plane, and draw some level curves for $f(x, y) = 2x + y$.

(b) Solve problem $(*)$ by a geometric argument.

(c) Write down the Kuhn–Tucker conditions. Verify that the point (x_0, y_0) you found in (b) satisfies the Kuhn–Tucker conditions.

(d) Suppose that $x^2 + (y+1)^2 \le 4$ in $(*)$ is replaced by $x^2 + (y+1)^2 \le 4.1$. Estimate the approximate change in the maximum value of $2x + y$.

Problem 3-08

Solve the problem

$$\max\left[\ln(x^2+2y)-\frac{1}{2}x^2-y\right]\quad \text{subject to}\quad y\geq 2/x,\;\; x\geq 1,\;\; y\geq 1$$

Problem 3-09

Consider the nonlinear programming problem

$$\text{maximize}\quad x^5-y^3 \qquad \text{subject to} \qquad x\leq 1, \quad x\leq y$$

(a) Find the only possible solution to this problem.

(b) Solve the problem by using iterated optimization: Find first the maximum value $f(x)$ in the problem to maximize x^5-y^3 s.t. $x\leq y$, where x is given and y varies. Then maximize $f(x)$ subject to $x\leq 1$.

Problem 3-10

Let f be a function of two variables given by

$$f(x,y)=-x^4-cx^2+6xy-6y^2$$

where c is a constant.

(a) For which values of c will f be concave in the whole plane?

Consider next the problem:

$$(*)\qquad \max\,(-x^4-y^4-4x^2+6xy-6y^2+ax+by)\quad \text{s.t.}\quad x+y^2\leq 1 \;\text{ and }\; y\geq -1.$$

Here are a and b constants.

(b) Write down the Kuhn–Tucker conditions for a point (x,y) to solve $(*)$.

(c) Find necessary and sufficient conditions on a and b for the maximum point in $(*)$ to be $(x,y)=(0,-1)$.

Problem 3-11

Let f and g be functions of two variables given by

$$f(x,y)=ax+by-6x^2-5xy-5y^2, \qquad g(x,y)=3-(x^2+y^2)^2$$

where a and b are constants.

(a) Show that f and g are both concave in the whole plane.

Next we study the problem:

$$(*)\qquad \max\,\big(f(x,y)+g(x,y)\big)\quad \text{s.t.}\quad x\geq 1 \quad\text{and}\quad y\leq \sqrt{x}$$

(b) Write down the Kuhn–Tucker conditions for a point (x,y) to solve $(*)$.

(c) Find necessary and sufficient conditions on a and b for the maximum point in $(*)$ to be $(x,y)=(1,0)$.

(d) Let $a=300$. For which value of b will $(*)$ have the solution $(x,y)=(4,2)$?

Problem 3-12

(a) Solve the nonlinear programming problem

$$\text{maximize } -\left(x+\frac{1}{2}\right)^2 - \frac{1}{2}y^2 \quad \text{subject to} \quad \begin{cases} y \geq e^{-x} \\ y \leq 2/3 \end{cases}$$

(*Hint:* You may need the fact that the equations $e^{-2x} = 2x+1$ and $y^2 = e^{1-y^2}$ have the solutions $x = 0$ and $y = \pm 1$, respectively.)

(b) Can you give a geometric interpretation of the problem?

Problem 3-13

Consider the problem

$$(*) \qquad \begin{aligned} &\text{maximize } \ a\ln(z+1) - z - 2x - y \\ &\text{s.t. } \ x \geq 0,\ y \geq 0,\ z \geq 0 \ \text{ and } \ z^2 \leq x + y \end{aligned}$$

where a is a positive constant.

(a) Write down the Kuhn-Tucker conditions for the solution of $(*)$.

(b) Find the solutions of the Kuhn-Tucker conditions. Consider the cases $a > 1$ and $a \leq 1$ separately.

(c) Prove that the points you found in (b) really solve problem $(*)$.

Problem 3-14

(a) Solve the problem minimize $[(x-2)^2 + (y-2)^2]$ s.t. $\begin{cases} x + y \leq 2, \\ x^2 - 4x + y \leq -2. \end{cases}$

(b) Can you give a geometric interpretation of the problem and thereby confirm the answer in (a)?

Problem 3-15

Consider the nonlinear programming problem

$$(\text{P}) \qquad \text{maximize } 4x + y \quad \text{s.t.} \quad \begin{cases} y - (x-1)^2 \leq 1 \\ y - (x-2)^2 \leq -1 \\ x \geq -2,\ y \geq 0,\ x \leq 2 \end{cases}$$

(a) Sketch the admissible region, and draw some level curves for the criterion function.

(b) Make use of the figure from (a) to show that $(1, 0)$ solves problem (P).

(c) Write down the necessary Kuhn–Tucker-conditions for the solution of (P), and show that $(1, 0)$ satisfies these conditions.

Problem 3-16

Solve the problem

$$\text{maximize} \quad (2x - x^2 + 2y - y^2 + z) \quad \text{s.t.} \quad \begin{cases} 1 - xy \le 0 \\ x + y + z \le 4 \\ x \ge 0,\ y \ge 0,\ z \ge 0 \end{cases}$$

Prove that you have really found the maximum.

Problem 3-17

Maximize $\ln(x + y + 3)$ subject to $x^2 + y^2 + z^2 \le 1,\ x + y + z \le 1.$

Problem 3-18

Solve the problem

$$\text{minimize} \ (x + y)^2 \quad \text{s.t.} \quad y \ge (x - 2)^2, \ xy \ge 1$$

(*Hint:* Use the fact that $x = 1$ is a solution of the equation $x(x - 2)^2 = 1$.)

Problem 3-19

Find the maximum of $xy + e^{x+z}$ subject to $0 \le x + z \le 1 - x^2 - 2y^2$.

Problem 3-20

Consider the problem

$$\text{maximize} \quad x + y - 1 \quad \text{s.t.} \quad \begin{cases} x + y + z^2 \le a \\ 3x^2 + 2y + \frac{1}{3}z \le 0 \end{cases}$$

Here a is a constant.

(a) Write down the Kuhn–Tucker conditions for a point (x, y, z) to solve the problem. Show that if (x, y, z) solves the problem, then $z \le 0$.

(b) Solve the problem when $a = 0$.

(c) Solve the problem when $a = 1$.

Problem 3-21

Consider the nonlinear programming problem

$$\text{maximize} \quad f(x, y) = (x - c)^{\alpha}(y - d)^{1-\alpha} \quad \text{subject to} \quad \begin{cases} (1) \ \ x \ge c \\ (2) \ \ y \ge d \\ (3) \ \ x + y \le 2 \\ (4) \ \ x \le c + d \end{cases} \qquad (*)$$

Here α, c, and d are constants with $\alpha \in (0, 1)$, $c \ge 0$, $d \ge 0$ and $2d < 2 - c$.

(a) Sketch the set S consisting of all (x, y) that satisfy (1)–(4). Explain why (1) and (2) cannot be binding at the optimum.

(b) Show that the problem is equivalent to

$$\text{maximize} \quad \big[\alpha \ln(x-c) + (1-\alpha)\ln(y-d)\big] \quad \text{subject to} \quad \begin{cases} (3) & x + y \le 2 \\ (4) & x \le c + d \end{cases} \qquad (**)$$

(c) Write down the Kuhn–Tucker-conditions for the solution to (**) and show that (3) must be binding at the optimum. Solve the problem.

Problem 3-22

Consider the problem

$$\max\,(4z - x^2 - y^2 - z^2) \quad \text{s.t.} \quad \begin{cases} z \le xy \\ x^2 + y^2 + z^2 \le 3 \end{cases}$$

(a) Write down the Kuhn–Tucker conditions for the problem.

(b) Verify that the problem has a solution, and find the solution.

(c) What is the approximate change in the maximum value of $4z - x^2 - y^2 - z^2$, if the first constraint is changed to $z \le xy + 0.1$?

Problem 3-23

(a) Find the maximum of xyz subject to $\begin{cases} x + y + z \le 5 \\ xy + xz + yz \le 8 \\ x \ge 0,\ y \ge 0,\ z \ge 0 \end{cases}$

(b) Prove that the problem in part (a) does not have a solution if we drop the nonnegativity conditions on x, y, and z.

4. Integration

Problem 4-01

Define the function F for all $T \ge 0$ by $F(T) = \int_0^T f(t)e^{-r(t-T)}\,dt$, where f is a given function and r is a given number. Find an expression for $F'(T)$ and show that $F'(T) - rF(T) = f(T)$.

Problem 4-02

Define the function F for all $x > 0$ by $F(x) = \displaystyle\int_0^x e^{xt^2}\,dt$.

(a) Find expressions for $F'(x)$ and $F''(x)$ by using Leibniz's formula.

(b) Show that F is strictly convex on $(0, \infty)$.

Problem 4-03

Define the function g by $g(x) = \displaystyle\int_\pi^{2\pi} \frac{\sin(xt)}{t}\,dt$ for all x. Show that $g'(1) = 0$.

Problem 4-04

Let $y(t)$ be defined by $y(t) = \int_0^t \sin^2(t+x)\,dx$. By differentiating $y(t)$ twice, show that $\ddot{y}(t) = 6\sin 2t\cos 2t + 2t - 4y(t)$.

Problem 4-05 Compute the integral $\int_0^1 \big(\int_0^1 xe^y\,dy\big)\,dx$.

Problem 4-06 Compute the integral $\int_0^1 \big(\int_0^4 (\sqrt{xy} + 2x + y)\,dy\big)\,dx$.

Problem 4-07 Compute the integral $\int_0^1 \big(\int_0^1 (x\sqrt{x^2+y}\,dx\big)\,dy$.

Problem 4-08

Compute the integral $\int_0^1 \big(\int_0^{\pi/2} xy^2\cos(x^2y)\,dy\big)\,dx$.

Problem 4-09

Compute the double integral $\int_\pi^{2\pi}\int_0^\pi \frac{x}{y^3}\cos\Big(\frac{x^2}{y}\Big)\,dx\,dy$.

Problem 4-10

Let $V(t) = \int_1^t \big(\int_1^t F(t,x,y)\,dx\big)\,dy$. Use Leibniz's formula to find an expression for $V'(t)$.

Problem 4-11

In a model by T. Haavelmo the following equations appear

$$\int_{t_1}^t z(\tau)\,d\tau = \int_{t_1+T(t_1)}^{t+T(t)} x(\tau)\,d\tau \quad \text{for all } t \geq t_1 \tag{$*$}$$

$$\int_t^{t+T(t)} \bar{y}(\tau)\,d\tau = 1 \quad \text{for all } t \geq t_1 \tag{$**$}$$

In both cases find an expression for dT/dt, assuming that $z(\tau)$, $x(\tau)$, $T(t)$, and $\bar{y}(\tau)$ are well-behaved and t_1 is a constant.

4145ex3 29.4.2008 621

5. Differential Equations of the First Order

Problem 5-01

Solve the differential equations

$$\text{(a) } \dot{x} + 4x = 3e^t, \quad x(0) = 1 \qquad\qquad \text{(b) } \dot{x} = \frac{e^{-3x}}{3 + \sqrt{t+8}}, \quad x(1) = 0$$

Problem 5-02

Solve the differential equation $\dot{x} = \dfrac{\sqrt[3]{ax+b}}{x}\, t^2$, where the constant a is $\neq 0$.

Problem 5-03

Find the solution of the following differential equation that satisfies the given initial condition:

$$\dot{x} = t(x-1)^2, \quad x(0) = 3$$

Problem 5-04

(a) Solve the differential equation $\dot{x} + 4x = 4e^{-2t}$, $x(0) = 1$.

(b) Suppose that $y = (a + \alpha k)\sqrt{t+1}$ denotes production as a function of capital k, where the factor $\sqrt{t+1}$ is due to technical progress. Suppose that a constant fraction $s \in (0,1)$ is saved, and that capital accumulation is equal to savings, so that we have the separable differential equation

$$\dot{k} = s(a + \alpha k)\sqrt{t+1}, \quad k(0) = k_0$$

The constants a, α and k_0 are positive. Find the solution.

Problem 5-05

(a) Find the general solution of the differential equation $\dot{x} + 2x = 2$.

(b) Find a function $w = w(t)$ such that

$$\ddot{w} + 2\dot{w} = 2, \quad w(0) = 0 \quad \text{and} \quad w\left(-\frac{1}{2}\right) = \frac{1}{2} - e.$$

Problem 5-06

In a growth model production Q is a function of capital K and labour L. Suppose that

(i) $\dot{K} = \gamma Q$ (investment is proportional to production)

(ii) $Q = K^\alpha L$

(iii) $\dot{L} = \beta$ (the rate of change of L with respect to t is constant)

Here γ, α, and β are positive constants, $\alpha < 1$.

(a) Derive a differential equation to determine K.

(b) Solve this equation when $K(0) = K_0$ and $L(0) = L_0$.

4145ex3 29.4.2008 621

Problem 5-07

Find the solution of the differential equation $\dot{x} + 3x = t^2$.

Problem 5-08

Determine the solution of the differential equation

$$x\frac{dx}{dt} = -\frac{1}{2}(x^2 - 25), \qquad x > 5 \tag{$*$}$$

that passes through the point $P = (0, 10)$. What is the slope of the solution curve at P? Show that every solution of $(*)$ is decreasing.

Problem 5-09

Find the solution of the differential equation $3x^2\dot{x} = (x^3+9)^{3/2}\ln t$ whose solution curve passes through the point $(t, x) = (1, 3)$.

Problem 5-10

Find the solution of the differential equation

$$\dot{x} = (2x + 1)^4 \cdot t^3 \cdot \sin(t^2)$$

whose integral curve passes through $(t, x) = (0, 0)$.

Problem 5-11

Find the solution of the differential equation

$$e^{2t}\dot{x} + e^{2t}(2 - 2t)x = \frac{e^{t^2+t}}{\sqrt{1 + e^t}}$$

whose integral curve passes through $(t_0, x_0) = (0, 3)$.

Problem 5-12

Find the solution of the differential equation

$$\dot{x} + 3t^2x = te^{t^2-t^3}$$

whose integral curve passes through $(t_0, x_0) = (-1, 0)$.

Problem 5-13

(a) Find the general solution of the differential equation $\dot{x} = \dfrac{2}{t+1}x$.

(b) Consider the differential equation

$$(t + 1)\dot{x} - 2(t + 1) = 2x + (t + 1)^5 \tag{$*$}$$

Introduce a new variable $u = u(t)$ by putting $x = (t+1)^2u$, and transform equation $(*)$ into a simple differential equation in the unknown function $u = u(t)$. Use this to find the general solution of $(*)$.

Problem 5-14

Find the general solution of the differential equation $\dot{x} = x^3 + 3x^2 - 2$. (*Hint:* Put $x = y + a$ and find a differential equation for y. Choose an a such that this equation becomes a Bernoulli equation.)

6. Differential Equations of the Second Order

Problem 6-01

Find the general solutions of the following differential equations:

$$\text{(a)} \quad \ddot{x} - 8\dot{x} + 17 = 0 \qquad\qquad \text{(b)} \quad \ddot{x} + 2\dot{x} + 5x = 0$$

Problem 6-02

Find the general solutions of the differential equations

$$\text{(a)} \quad \ddot{x} + \frac{7}{2}\dot{x} - 2x = 0 \qquad \text{(b)} \quad \ddot{x} + \frac{7}{2}\dot{x} - 2x = t + \sin t$$

(*Hint:* (b) has a particular solution of the form $u^* = At + B + C\sin t + D\cos t$.)

Problem 6-03

Find the general solutions of the following differential equations:

$$\text{(i)} \ 4\ddot{x} - 15\dot{x} + 14x = 0 \qquad\qquad \text{(ii)} \ 4\ddot{x} - 15\dot{x} + 14x = t + \sin t$$

Problem 6-04

Find the general solution of $\ddot{x} - 6\dot{x} + 25x = t$.

Problem 6-05

Find the general solution of the differential equation $3\ddot{x} + 10\dot{x} + 3x = f(t)$ in the following three cases:

$$\text{(i)} \ f(t) = 0, \qquad \text{(ii)} \ f(t) = 8e^{-t} + 6, \qquad \text{(iii)} \ f(t) = -8e^{-3t}.$$

Problem 6-06

Find the general solution of the differential equation $\ddot{x} + \dot{x} + 2x = t^2 + 2$.

Problem 6-07

Solve the differential equations

$$\ddot{x} - a\dot{x} + (a-2)x = 0 \qquad \text{(i)}$$
$$\ddot{x} - a\dot{x} + (a-2)x = t \qquad (a \neq 2) \qquad \text{(ii)}$$

Problem 6-08

In connection with a problem in utility theory the following differential equations are encountered:

$$\text{(i)}\ g''(t) = -\frac{1}{4}\,g'(t) \qquad \text{(ii)}\ g''(t) = -\frac{2}{t+1}\,g'(t)$$

Find the general solutions of these equations.

Problem 6-09

In an economic model one encounters the differential equation

$$\ddot{Y} + (\alpha l + \beta)\dot{Y} + \alpha\beta(l+m)Y = -\alpha\beta t - \frac{\alpha l + \beta}{l+m} \qquad (*)$$

where α, β, l and m are positive constants.

(a) Find a particular solution of $(*)$.

(b) Put $\alpha = 1/4$, $\beta = 3/4$, $l = 1$ and $m = 17/3$, and find the general solution to the equation in this case.

(c) Discuss conditions that ensure that the solutions of $(*)$ give oscillations. What can you then say about the behaviour of the solutions as t approaches infinity?

Problem 6-10

A model describing the market behaviour of firms includes the differential equation

$$\tfrac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - \rho V(x) = w - x \qquad (x > 0) \qquad (*)$$

Here σ, μ, ρ, and w are positive constants, $\rho \neq \mu$, while $V(x)$ is an unknown function.

(a) Show that the homogeneous equation corresponding to $(*)$ has solutions of the form $V(x) = x^a$. Find the general solution of the homogeneous equation.

(b) Find a particular solution of $(*)$.

(c) Find the general solution of the equation $x^2V''(x) + xV'(x) - 4V(x) = 10 - x$.

Problem 6-11

Consider the differential equation

$$\ddot{x} - 2(k-1)\dot{x} + (k^2 - 4)x = 2e^{(4-k)t} \qquad (*)$$

where k is a real number.

(a) Find the general solution of the homogeneous equation corresponding to $(*)$ for all values of k.

(b) Find a particular solution of $(*)$ for each value of k.

(c) Let $k = 3$. Find the integral curve for $(*)$ that passes through the origin and is tangent to the t-axis at that point. Is $t = 0$ a local extreme point for the corresponding function?

4145ex3 29.4.2008 621

Problem 6-12

In an economic problem the price function $p = p(t)$ satisfies the equation

$$\dot{p}(t) = \beta \int_{-\infty}^{t} \big[D(p(\tau)) - S(p(\tau))\big] e^{-\alpha(t-\tau)}\, d\tau$$

where $D(p) = a - bp$ is a demand function, $S(p) = -c + dp$ is a supply function, and α, β, a, b, c, and d are positive constants.

(a) Show that p satisfies the differential equation

$$\ddot{p} + \alpha\dot{p} + \beta(b+d)p = \beta(a+c) \tag{1}$$

(b) Determine the equilibrium value p^* for (1), and find the general solution of (1).

(c) Show that (1) is stable. Decide for what values of the parameters the solution exhibits damped oscillations about p^*.

Problem 6-13

Given the second-order differential equation

$$t^2\ddot{x} + t\dot{x} - x = 0, \qquad t > 0 \tag{$*$}$$

(a) Introduce the substitution $z = tx$ and derive a second-order differential equation for z. Solve this equation and find the general solution of $(*)$.

(b) Find the solution of $(*)$ with $x(1) = 1$ and $\dot{x}(1) = 1$.

Problem 6-14

Solve the differential equation $\ddot{x} + 10\dot{x} + 25x = 5e^{kt} + \sin t$ for all values of k.

Problem 6-15

(a) Find the general solution of the differential equation $2\ddot{x} + 8\dot{x} + 26x = e^{2t}$.

(b) Explain how to find the general solution if e^{2t} is replaced by $\sin 3t$.

7. Systems of Differential Equations

Problem 7-01

Given the following system of differential equations:

$$\begin{aligned} \dot{x} &= x + y + t \\ \dot{y} &= -x + 2y \end{aligned}$$

Deduce a second-order differential equation for x. Solve this equation and then find the general solution $(x(t), y(t))$ of the system.

4145ex3 29.4.2008 621

Problem 7-02

Consider the differential equation system

$$\begin{aligned} \dot{x} &= ax + 2y + \alpha \\ \dot{y} &= 2x + ay + \beta \end{aligned} \tag{$*$}$$

where a, α and β are constants.

(a) Find the general solution of $(*)$ for all values of the constants.

(b) Assuming that $a^2 \neq 4$, find the equilibrium point. Find sufficient conditions for $(*)$ (i) to be locally asymptotically stable, (ii) to have a saddle point equilibrium.

(c) Let $a = -1$, $\alpha = -4$ and $\beta = -1$. Draw a phase diagram of the system $(*)$, and determine a solution curve that converges to the equilibrium point.

Problem 7-03

(a) Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= -x^2 - x - y \\ \dot{y} &= -y^2 - 2xy \end{aligned} \tag{S}$$

The points $(-1, 0)$, $(0, 0)$ and $(1, -2)$ are equilibrium points for the system. Try to decide if they are locally asymptotically stable or saddle points.

(b) The system (S) has a solution $(x(t), y(t))$ with $x(0) = 1$ and $y(0) = -1$ and such that $x(t) + y(t)$ is a constant. Find this solution, and draw a sketch that indicates how $(x(t), y(t))$ traces out a curve in the xy-plane as t runs through $[0, \infty)$.

Problem 7-04

Consider the differential equation system

$$\begin{aligned} \dot{x} &= y - x^2 - xy \\ \dot{y} &= x - y^2 - xy \end{aligned} \tag{$*$}$$

(a) Find the equilibrium points. If possible, decide if they are locally asymptotically stable or saddle points.

(b) Put $z = x + y$ and find a differential equation for z. Find the general solution of this equation.

(c) Find solutions of $(*)$ through $(1, 1)$, $(1/4, 1/4)$, and $(-1, -1)$, respectively. (*Hint:* Try to find a differential equation for $w = x - y$.)

Problem 7-05

Consider the differential equation system

$$\dot{x} = \tfrac{1}{2}x^3 - y, \qquad \dot{y} = 2x - y$$

(a) Find all equilibrium points and classify each of them (if possible).

(b) Draw a phase diagram and indicate some possible integral curves.

(c) The system has a saddle point in the first quadrant. What is the limit of the slope of the two integral curves that converge towards this saddle point?

Problem 7-06

Consider the differential equation system

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -xy - y^2 \end{aligned} \qquad (*)$$

(a) Draw a phase diagram and draw some typical solution curves.
(b) Find the equilibrium points and classify them, if possible.
(c) Solve the equation system $(*)$ with $x(0) = -1$, $y(0) = 1$. (One of the integrals cannot be evaluated.) Find $\lim_{t\to\infty}(x(t), y(t))$.

8. Calculus of Variations

Problem 8-01

Consider the variational problem

$$\max \int_0^1 (2xe^{-t} - 2x\dot{x} - \dot{x}^2)\, dt, \qquad x(0) = 0,\ x(1) = 1$$

(a) Write down the Euler equation for the problem.
(b) Find the solution of the problem, assuming it has one.

Problem 8-02

Solve the variational problem

$$\max \int_0^1 (4xe^{-t} - 5x^2 - \dot{x}^2)e^{-4t}\, dt, \qquad x(0) = 5/3,\ \ x(1) = 2e^{-1}$$

Problem 8-03

Consider the variational problem

$$\max \int_0^T \left(\frac{1}{100}tx - \dot{x}^2\right) e^{-t/10}\, dt, \quad x(0) = 0, \quad x(T) = S$$

(a) Find the Euler equation and its general solution.
(b) Put $T = 10$ and $S = 20$ and find the solution to the problem in this case. Show that you really have found the optimal solution.

Problem 8-04

Consider the variational problem

$$\max \int_0^T (a - bK^2 - cK\dot{K} - d\dot{K}^2)e^{-rt}\, dt, \qquad K(0) = 0,\ K(T) \geq K_T$$

where a, b, c, d, r, T, and K_T are positive constants.
(a) Find the Euler equation.

(b) Find a necessary and sufficient condition for $F(t, K, \dot{K}) = (a - bK^2 - cK\dot{K} - d\dot{K}^2)e^{-rt}$ to be concave in $(K, \dot{K})$.

(c) Let $a = 100$, $b = c = 1$, $d = 2$, $r = 1$, $T = 10$, and $K_{10} = 4$. Verify that the Euler equation is $\ddot{K} - \dot{K} - \frac{3}{4}K = 0$. Find the only solution of the Euler equation that satisfies the boundary conditions. Why is this the optimal solution to the variational problem?

Problem 8-05

Consider the variational problem

$$\min \int_1^2 (x^2 + tx\dot{x} + t^2\dot{x}^2)\, dt, \qquad x(1) = 0, \quad x(2) = 1$$

(a) Find the Euler equation.

(b) The Euler equation has two solutions of the form $x = t^a$, for suitable values of the constant a. Use this information to solve the problem.

Problem 8-06

Consider the variational problem

$$\min \int_0^T \left[px^2 + q\left(\frac{1}{b}(\dot{x} - ax)\right)^2\right] dt, \qquad x(0) = x_0, \ \ x(T) = x_T$$

where p, q, a, b, T, x_0, and x_T are nonnegative constants, $b \neq 0$, $q \neq 0$.

(a) Write down the Euler equation for the problem and find its general solution.

(b) Choose $p = 0$, $q = 1$, $a = 1$, $b = 1$, $T = 1$, $x_0 = 0$ and $x_T = 1$. Find the solution of the problem in this case.

Problem 8-07

(a) Find the Euler equation for the following variational problem:

$$\min \int_{t_0}^{t_1} (b(t)x + a(t)\dot{x}^2)\, dt, \qquad x(t_0) = x_0, \ \ x(t_1) = x_1$$

Here t_0, t_1, x_0, and x_1 are constants, while $a(t)$ and $b(t)$ are given positive, differentiable functions.

(b) Show that the general solution of the Euler equation can be written in the form

$$x(t) = \int \left(\frac{C}{a(t)} + \frac{1}{2a(t)} \int b(t)\, dt \right) dt + D,$$

where C and D are arbitrary constants.

(c) Find $x(t)$ if $a(t) = t$, $b(t) = t^2$, $t_0 = 1$, $t_1 = 3$, $x(1) = 0$, and $x(3) = 2$.

4145ex3 29.4.2008 621

Problem 8-08

(a) Consider the variational problem

$$\max \int_0^1 (-2\dot{x} - \dot{x}^2)e^{-t/10}\, dt, \qquad x(0) = 1,\ x(1) = 0 \tag{1}$$

Find the associated Euler equation, and find the general solution of that equation. Then solve problem (1).

(b) At time $t = 0$ a certain oil field contains $\bar{x}$ barrels of oil. It is desired to extract all of the oil during a given time interval $[0, T]$. If $x(t)$ is the number of barrels of oil left at time t, then $-\dot{x}$ is the extraction rate (which is ≥ 0 when $x(t)$ is decreasing). We assume that the world market price per barrel of oil is given and equal to $ae^{\alpha t}$. The extraction costs per unit of time are assumed to be $(\dot{x}(t))^2 e^{\beta t}$. The profit per unit of time is then $\pi = -\dot{x}(t)ae^{\alpha t} - (\dot{x}(t))^2 e^{\beta t}$. Here a, α, and β are constants, $a > 0$. This leads to the variational problem

$$\max \int_0^T \bigl[-\dot{x}(t)ae^{\alpha t} - (\dot{x}(t))^2 e^{\beta t}\bigr] e^{-rt}\, dt, \qquad x(0) = \bar{x},\ x(T) = 0 \tag{2}$$

where r is a positive constant. Find the Euler equation for problem (2), and show that at the optimum $\partial\pi/\partial\dot{x} = ce^{rt}$ for some constant c.

Problem 8-09

Consider the variational problem

$$\min \int_0^1 (x^2 + 2xt\dot{x} + \dot{x}^2)\, dt, \qquad x(0) = 1,\ x(1) = 1$$

(a) Find the Euler equation for the problem, and the only admissible solution of it.

(b) Show that

$$\int_0^1 \bigl[(x(t))^2 + 2x(t)\cdot t\cdot \dot{x}(t) + (\dot{x}(t))^2\bigr]\, dt = 1 + \int_0^1 (\dot{x}(t))^2\, dt$$

for all admissible functions $x(t)$ in the problem. (*Hint:* $\dfrac{d}{dt}(tx^2) = x^2 + 2tx\dot{x}$.)

(c) Can we conclude from (b) that the function found in (a) solves the problem?

Problem 8-10

Consider the variational problem

$$\text{minimize} \int_0^T e^{-rt}\bigl[g(\dot{x}) + c(t)x\bigr]\, dt, \qquad x(0) = 0,\ x(T) = B$$

where g and c are given functions, T, r and B are given positive numbers, and $x = x(t)$ is the unknown function.

(a) Write down the Euler equation for this problem.

(b) Find the solution of the problem if $r > 0$, $g(\dot{x}) = \dot{x}^2$, $c(t) = 2$.

(c) The problem above can be given the following interpretation: One wants to produce B units of a product during the time interval $[0, T]$. Production per unit of time is $\dot{x}$, $g(\dot{x})$ is the production cost per unit of time, $c(t)$ is the storage cost per unit of time and unit of product, and r is the interest rate. This interpretation works well only if the solution of the problem has $\dot{x}(t) \geq 0$ in $[0, T]$. Find necessary and sufficient conditions on the parameters for $\dot{x}(t)$ to be nonnegative for all t in $[0, T]$ in the case in part (b).

9. Control Theory

Problem 9-01

Solve the control problem

$$\max \int_0^2 (2x - 3u - \alpha u^2)\, dt, \qquad \dot{x} = x + u,\ \ x(0) = 5,\ \ x(2) \text{ free},\ \ u \in (-\infty, \infty)$$

where α is a positive constant.

Problem 9-02

Consider the control problem

$$\max_{u(t)\in\mathbb{R}} \int_0^T -[x(t) - u(t) + 2]^2 e^{-rt}\, dt, \qquad \dot{x} = u(t) - \delta x(t),\ \ x(0) = x_0,\ \ x(T) = x_T$$

Here T, r, δ, x_0, and x_T are fixed positive numbers.

(a) Write down the conditions of the maximum principle.

(b) Show that the Hamiltonian function is concave in (x, u).

(c) Solve the problem with $T = 10$, $r = 0.1$, $\delta = 0.5$, $x_0 = 0$, and $x_{10} = 8$.

Problem 9-03

(a) Solve the optimal control problem

$$\max \int_0^2 \big(x(t) - (u(t))^2\big)\, dt, \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 0,\ x(2) \text{ free},\ u(t) \in (-\infty, \infty)$$

(b) What is the optimal solution $x^*(t)$ if we require that $u(t) \in [0, 1]$ for all t in $[0, 2]$?

Problem 9-04

(a) Solve the control problem

$$\max \int_0^5 -3u^2 e^{-0.15t}\, dt, \qquad \dot{x} = u,\ \ x(0) = 0,\ \ x(5) = 1500,\ \ u \geq 0$$

(b) A municipality wishes to cultivate a plot over a period of 5 years. Let $x(t)$ be the number of acres cultivated up to time t and let $u(t)$ be the rate of cultivation, so that $\dot{x}(t) = u(t)$. Let the cultivation cost per unit of time be given by the function

$C(u,t)$. If the interest rate is r, the total discounted cost of cultivation over the period from $t=0$ to $t=5$ is $\int_0^5 C(u,t)e^{-rt}\,dt$. Consider the problem

$$\min \int_0^5 C(u,t)e^{-rt}\,dt, \qquad \dot{x}=u,\ \ x(0)=0,\ \ x(5)\geq 1500,\ \ u\geq 0$$

Write down the conditions given by the maximum principle.

(c) Solve the problem when $r=0$ and $C(u,t)=g(u)$, with $g(0)=0$, $g(u)\geq 0$ and $g''(u)>0$

Problem 9-05

Consider the variational problem

$$\max \int_0^T (ax^2+2bx\dot{x}+c\dot{x}^2+dt^2\dot{x})e^{-rt}\,dt, \qquad x(0)=x_0, \quad x(T)=x_T \qquad (*)$$

(a) For what values of the constants a, b, c, d, and r is $(ax^2+2bxy+cy^2+dt^2y)e^{-rt}$ concave with respect to (x,y)?

(b) Find the Euler equation associated with $(*)$.

(c) Solve the problem

$$\max \int_0^1 (-9x^2+2x\dot{x}-\dot{x}^2+3t^2\dot{x})\,dt, \qquad x(0)=0, \quad x(1)=0 \qquad (**)$$

(You can use the result in (b).)

(d) Transform the problem $(**)$ in (c) into a control problem and find the optimal solution when the terminal condition is changed from $x(1)=0$ to

(i) $x(1)$ free, (ii) $x(1)\geq 2$.

Problem 9-06

Consider the control problem (T is a fixed positive number)

$$\max \int_0^T \Big(2x^2-\frac{1}{2}u^2\Big)\,dt, \qquad \dot{x}=u, \quad x(0)=1, \quad x(T)\ \text{free}, \quad u(t)\in(-\infty,\infty)$$

(a) Write down the conditions given by the maximum principle for this problem.

(b) Assume that $T=\pi/8$, and find the only possible optimal solution. (Take for granted that an optimal solution exists.)

(c) Find the only possible solution of the necessary conditions when $T=\pi+\pi/8$. Is this solution optimal? (*Hint:* Consider the control function $u(t)=c$ if $t\in[0,1]$, $u(t)=0$ if $t\in(1,\pi+\pi/8]$.)

Problem 9-07

Consider the optimal control problem

$$\text{maximize} \int_0^T (1-u)x^2\,dt \quad \text{s.t.} \quad \dot{x}=ux,\ \ u\in[0,1],\ \ x(0)=1,\ \ x(T)\ \text{is free} \qquad (*)$$

(a) Write down the conditions that the maximum principle gives for an admissible pair $(x^*(t),u^*(t))$ to be a solution of problem $(*)$.

(b) Show that the adjoint function $p(t)$ must be strictly decreasing.

(c) It can be shown (but you are not supposed to do so) that $(*)$ has an optimal solution. Find this solution when $T > 1/2$.

Problem 9-08

In growth theory we encounter the optimal control problem

$$\max_{I\in\mathbb{R}} \int_0^T (aK - bK^2 - cI^2)e^{-rt}\,dt, \qquad \dot{K} = I - \delta K,\ \ K(0) = K_0,\ \ K(T) \text{ free}$$

where all the parameters are positive, K (capital) is the state variable, and I (investment) is the control variable.

(a) Write down the conditions in the maximum principle for a pair $(K^*(t), I^*(t))$ to solve the problem. Use the "current-value" formulation.

(b) Deduce a second-order differential equation for the optimal state variable.

(c) Show that when $a = 12$, $b = 0.256$, $c = 10$, $r = 0.2$, and $\delta = 0.02$, the differential equation in (b) reduces to

$$\ddot{K}^* - 0.2\dot{K}^* - 0.03K^* = -0.6$$

Solve the control problem in this case with $K_0 = 0$ and $T = 10$.

(d) With the choice of parameters in (c), replace the objective function by

$$\int_0^{10} (12K - 0.256K^2 - 10I^2)e^{-0.2t}\,dt + K(10)$$

Explain how to find the solution in this case. In particular, find the transversality condition. You are not required to determine the constants.

Problem 9-09

Consider the control problem

$$\max \int_0^T (x - u)\,dt, \qquad \dot{x} = aue^{-2t} - x,\ \ x(0) = x_0,\ \ x(T) \text{ is free},\ \ u \in [0, 1]$$

where T, a, and x_0 are positive constants.

(a) Write down the conditions given by the maximum principle. Find an explicit expression for the adjoint function $p(t)$, and determine the possible values of an optimal control.

(b) Put $T = \ln 10$, $a = 5$ and $x_0 = 5$ and solve the problem in this case.

(c) What is the solution if $T = \ln 10$, $a = 1/2$ and $x_0 = 5$?

Problem 9-10

Consider the control problem (a is a given constant)

$$\max \int_0^1 -(x - u - a)^2\,dt, \qquad \dot{x} = u - x,\ \ x(0) = 1,\ \ x(1) \text{ free},\ \ u \in [0, \infty)$$

(a) Put $a = 4$. Show that $u^*(t) = 0$ for all t is an optimal control by showing that all the conditions in Mangasarian's sufficiency theorem are satisfied.

(b) Put $a = 1/3$. Find the optimal control in this case. (*Hint:* Try $u^*(t) > 0$ for all t.)

Problem 9-11

Solve the control problem

$$\text{maximize} \int_0^2 (x - \tfrac{1}{2}u)\, dt, \qquad u \in [0, 1],$$
$$\dot{x} = u, \quad \dot{y} = u, \quad x(0) = 0, \quad y(0) = 0, \quad x(2) \text{ free}, \quad y(2) \le 1$$

Here u is the control variable.

Problem 9-12

Find the only possible solution to the control problem

$$\max \int_0^2 (u^2 - x)\, dt \quad \text{s.t.} \quad \dot{x} = u, \ \ x(0) = 0, \ \ x(2) \text{ is free}, \ \ 0 \le u \le 1$$

that is, find the only admissible pair $(x(t), u(t))$ that satisfies the conditions in the maximum principle.

Problem 9-13

Consider the control problem (T is a given positive constant)

$$\max \int_0^T (x^2 - x)\, dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(T) \text{ free}, \quad u = u(t) \in [0, 1]$$

(a) Write down the conditions given by the maximum principle.

(b) Prove that $p(t)$ is concave. Study possible behaviours of $p(t)$. Assuming that an optimal solution exists, find it.

Problem 9-14

Consider the control problem

$$\max \int_0^T (ax - bu)\, dt, \ \ \dot{x} = x + u, \ \ x(0) = x_0, \ \ x(T) \text{ free}, \ \ u \in [0, 2]$$

where all the constants are positive.

(a) Solve the problem. (*Hint:* You will have to distinguish between the cases $b < a(e^T - 1)$ and $b \ge a(e^T - 1)$.)

(b) Let $J(x_0, T)$ denote the optimal value function. Show that $\partial J(x_0, T)/\partial x_0 = p(0)$, where $p(t)$ is the adjoint function, and that $\partial J(x_0, T)/\partial T = H^*(T)$, when we let $H^*(t)$ denote the Hamiltonian evaluated "along" the optimal path.

Problem 9-15

Consider the control problem

$$\text{maximize} \int_0^T (24x - ux)\, dt \quad \text{s.t.} \quad \dot{x} = ux^3, \ \ u \in [0, 1], \ \ x(0) = 1, \ \ x(T) \text{ free}$$

Find the only possible solution of the problem when $T = 1/3$. (*Hint:* Show that $p(t)(x^*(t))^2$ must be strictly decreasing.)

Problem 9-16

Find the only possible solution of the control problem

$$\max \int_0^2 x^3 \, dt, \qquad \dot{x} = u \in [-1, 1], \quad x(0) = 0, \quad x(2) = 0$$

Problem 9-17

(a) Solve the optimal control problem

$$\max \int_0^1 (x + u) \, dt, \qquad \dot{x} = 1 - \tfrac{1}{2}u^2, \;\; x(0) = 0, \;\; x(1) \geq 0, \;\; u \in (-\infty, \infty)$$

(b) (Difficult.) Solve the problem if we require that $u \geq 1$.

10. Difference Equations

Problem 10-01

Find the general solution of the difference equation $x_{t+2} + 6x_{t+1} + 10x_t = 0$.

Problem 10-02

Find the general solution of the difference equation $x_{t+2} - 6x_{t+1} + 25x_t = 1$.

Problem 10-03

Find the general solution of the difference equation $x_{t+2} + x_{t+1} - 6x_t = 5^t + t$.

Problem 10-04

Find the solution of the difference equation $x_{t+2} + 4x_{t+1} - 12x_t = 7t^2 + 2t - 6$ that satisfies $x_0 = -3$ and $x_1 = 9$.

Problem 10-05

(a) Solve the difference equation

$$x_{t+2} - \frac{5}{2}x_{t+1} + x_t = 10 \cdot 3^t, \qquad t = 0, 1, 2, \ldots$$

Determine the solution that gives $x_0 = 0$, $x_1 = 2$.

(b) The following difference equation appears in dynamic consumer theory:

$$-\alpha x_{t+1} + (1 + \alpha^2)x_t - \alpha x_{t-1} = K\beta^t, \qquad t = 1, 2, \ldots$$

Here α, K and β are constants, α and β positive. Determine the general solution of the equation when $\alpha \neq 1$, $\beta \neq \alpha$ and $\beta \neq 1/\alpha$.

Problem 10-06

(a) Solve the difference equation $x_{t+2} - x_{t+1} - 6x_t = c$ with $x_0 = 1$ and $x_1 = 1$ for $c = 0$ and for $c = 1$.

(b) Consider the following system of difference equations:

$$\begin{aligned} x_{t+1} &= x_t + 2y_t \\ y_{t+1} &= 3x_t \end{aligned} \qquad (*)$$

where $t = 0, 1, 2, \ldots, x_0 = 1, y_0 = 0$. Derive a second-order difference equation for x_t, and solve this equation and the system.

11. Dynamic Programming

Problem 11-01

Consider the dynamic programming problem

$$\max \left\{ \sum_{t=0}^{T-1} (-u_t^2) - x_T^2 \right\} \quad \text{subject to} \quad x_{t+1} = x_t + u_t \quad u_t \in (-\infty, \infty)$$

(a) Let $J_s(x)$ be the value function, and find $J_T(x)$, $u_T^*(x)$, $J_{T-1}(x)$, $u_{T-1}^*(x)$, $J_{T-2}(x)$, and $u_{T-2}^*(x)$.

(b) Find general expressions for $J_{T-k}(x)$ and $u_{T-k}^*(x)$.

Problem 11-02

Consider the problem

$$\max \sum_{t=0}^{T} (x_t - u_t) \quad \text{subject to} \quad x_{t+1} = x_t + u_t, \quad u_t \in [0, 1]$$

(a) Find the optimal control functions $u_t^*(x)$ and the corresponding value functions $J_t(x)$ for $t = T,\ t = T-1,\ \ldots,\ t = T-4$.

(b) Try to find expressions for $u_{T-k}^*(x)$ and $J_{T-k}(x)$ for all $k = 0, 1, 2, \ldots, T$.

Problem 11-03

Consider the problem

$$\max \sum_{t=0}^{T-1} \ln u_t + \ln x_T \quad \text{subject to} \quad x_{t+1} = x_t - u_t,\ \ x_0 > 0,\ \ u_t \in (0, x_t)$$

(a) Find $J_s(x)$ and $u_s^*(x)$ for $s = T,\ T-1,\ T-2$.

(b) Prove that $J_{T-k}(x) = (k+1) \ln \dfrac{x}{k+1}$ for all $k = 0, 1, \ldots, T$, and find the associated optimal control.

Problem 11-04

Consider the problem

$$\max \sum_{t=0}^{T} x_t^2(1+u_t) \quad \text{subject to} \quad x_{t+1} = x_t(1-u_t), \quad u_t \in [0,1]$$

with x_0 given.

(a) Find $J_T(x)$ and $J_{T-1}(x)$ and the corresponding optimal controls $u_T^*(x)$ and $u_{T-1}^*(x)$.

(b) Show by induction that $J_{T-n}(x) = (n+2)x^2$ for $n = 0, 1, 2, \ldots, T$. Find the optimal pair (x_t^*, u_t^*) for the problem and the maximum value of the objective function.

Problem 11-05

Consider the dynamic programming problem

$$\max \sum_{t=0}^{T-1} \sqrt{u_t} - x_T \quad \text{subject to} \quad x_{t+1} = 2(x_t + u_t), \quad u_t \in (0, \infty)$$

(a) Find $J_T(x)$, $J_{T-1}(x)$, and $J_{T-2}(x)$.

(b) Find $J_t(x)$ for all $t = 1, \ldots, T$. Find the optimal pair (x_t^*, u_t^*) for the problem and the maximum value of the objective function.

Problem 11-06

Consider the dynamic programming problem

$$\max \sum_{t=0}^{T} (x_t + \ln u_t) \quad \text{when} \quad x_{t+1} = x_t - u_t, \;\; u_t \in (0,1], \;\; x_0 \text{ given},$$

(a) Find $J_T(x)$, $u_T^*(x)$, $J_{T-1}(x)$, $u_{T-1}^*(x)$, $J_{T-2}(x)$, and $u_{T-2}^*(x)$.

(b) Prove by induction that $J_{T-k}(x) = (t+1)x - t - \ln(1 \cdot 2 \cdot 3 \cdots t)$, $t = 1, 2, \ldots, T$.

Problem 11-07

Consider the dynamic programming problem

$$\max \sum_{t=0}^{T-1} 2\sqrt{u_t x_t} + \sqrt{x_T} \quad \text{subject to} \quad x_{t+1} = x_t - u_t x_t, \quad u_t \in [0,1]$$

(Interpretation: My wealth today is $x_0 \geq 0$. Every day, i.e. for every $t < T$, I spend $u_t x_t$ dollars on consumption. On day T I spend the remaining wealth, x_T.

(a) Find $J_{T-1}(x)$ and $J_{T-2}(x)$ and the corresponding controls $u_{T-1}^*(x)$ and $u_{T-2}^*(x)$.

(b) Show that $J_t^*(x)$ can be written in the form $J_t^*(x) = k_t\sqrt{x}$, and find a difference-equation for k_t.

Answers

1-01. (a) $\mathbf{A}^2 = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 13 & 14 \\ 21 & 6 \end{pmatrix}$ (b) $\lambda_1{=}3$, $\mathbf{v}_1{=}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2{=}-2$, $\mathbf{v}_2{=}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(c) $\mathbf{P}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$

1-02. (a) $\lambda_1 = 2a$, $\lambda_2 = 1 + \sqrt{1-a}$, $\lambda_3 = 1 - \sqrt{1-a}$.

(b) (i) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_{2,3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$; (ii) $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$.

1-03. (a) No, $\mathbf{a} - \mathbf{c} = (\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c})$. (b) No.

1-04. (a) $\lambda_1 = 3$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) With $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_0 = \frac{3}{2}\mathbf{u} - \frac{1}{2}\mathbf{v}$, $\mathbf{x}_t = \frac{1}{2}\begin{pmatrix} 3^{t+1} + (-1)^{t+1} \\ 3^{t+1} - (-1)^{t+1} \end{pmatrix}$.

1-05. If $a = 1$, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. The eigenvectors are $s(0, -1, 1)'$ and $t(1, -1, 2)'$, with $s \neq 0$, $t \neq 0$. (In this case it is not possible to find three linearly independent eigenvectors.)
If $a = -3$, the eigenvalues are $\lambda_1 = -6$, $\lambda_2 = -1$, $\lambda_3 = 3$. Eigenvectors $r(-3, -1, 2)'$, $s(0, -3, 1)'$, $t(0, 1, 1)'$, with $r \neq 0$, $s \neq 0$, $t \neq 0$. (When an $n \times n$ matrix has n distinct eigenvalues, one can always find n linearly independent eigenvectors.)

1-06. (a) $r(\mathbf{D}_t) = \begin{cases} 4 & \text{if } t \neq 0 \text{ and } t \neq 1 \\ 3 & \text{if } t = 0 \text{ or } t = 1 \end{cases}$ (b) $\mathbf{X} = \mathbf{C}^{-1} - \mathbf{BA}$

1-07. (a) $\mathbf{A}^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} \|\mathbf{b}_1\|^2 & 0 & 0 \\ 0 & \|\mathbf{b}_2\|^2 & 0 \\ 0 & 0 & \|\mathbf{b}_3\|^2 \end{pmatrix}$

(c) $\mathbf{B}^{-1} = \mathbf{A}^{-1}\mathbf{B}'$ (d) $\mathbf{P}^{-1} = \frac{1}{81}\mathbf{P}$

1-08. (a) $a = -5$, $b = 4$. The eigenvalues are $\lambda = 4$ (multiplicity 2) and $\lambda = 1$.
(b) Direct verification.

(c) $\mathbf{C}^{-1} = \mathbf{C}'$, $\mathbf{C}^{-1}\mathbf{AC} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\mathbf{B}^2 = \mathbf{A}$ when $d_1^2 = d_2^2 = 4$ and $d_3^2 = 1$,
so one can choose, for example, $d_1 = d_2 = 2$ and $d_3 = 1$.

1-09. (a) 3 (b) Linearly independent in the following three cases: (i) $x \neq 0$ and $z \neq 4$, (ii) $z = 4$ and $xy \neq 1$, (iii) $x = 0$ and $z \neq 2$.

1-10. (a) $r(\mathbf{A}_k) = 3$ for $k \neq -1$; $r(\mathbf{A}_k) = 2$ for $k = -1$ (b) The characteristic equation: $(1 - \lambda)(\lambda^2 + \lambda - 3(1 + k)) = 0$. All roots are real $\iff k \geq -13/12$.

(c) $\mathbf{P}'\mathbf{A}_3\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

1-11. (a) $(*)$ always has solutions. There are 2 degrees of freedom.
(b) The new system has solutions $\iff r(\tilde{\mathbf{A}}) = r(\tilde{\mathbf{A}}_c)$, where

$$\tilde{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{15} & 0 \\ a_{21} & a_{22} & \dots & a_{25} & 0 \\ a_{31} & a_{32} & \dots & a_{35} & 0 \\ a_{41} & a_{42} & \dots & a_{45} & a_{46} \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{A}}_c = \begin{pmatrix} & c_1 \\ \tilde{\mathbf{A}} & c_2 \\ & c_3 \\ & c_4 \end{pmatrix}$$

If $a_{46} \neq 0$, the system has solutions with 2 degrees of freedom. If $a_{46} = 0$, the following holds: If $r(\tilde{\mathbf{A}}) = 4$, then the system has solutions with 2 degrees of freedom (remember that x_6 is an unknown!). If $r(\tilde{\mathbf{A}}) = 3$, then the system has solutions if $r(\tilde{\mathbf{A}}_c) = 3$ too. The number of degrees of freedom is then 2. If $r(\tilde{\mathbf{A}}) = 3$ and $r(\tilde{\mathbf{A}}_c) = 4$, there are no solutions.

1-12. $\mathbf{D}(s)$ has rank 4 $\iff 9s^3 + 16s^2 - 15s - 10 \neq 0,\ r(\mathbf{D}(1)) = 3$.
(b) 1 degree of freedom.

1-13. (a) $\mathbf{X}'\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{X}$ (b) $\mathbf{X} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}'$

1-14. (a) $r(\mathbf{A}) = 2$, $(\mathbf{A}\mathbf{A}')^{-1} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$ (b) $\mathbf{C} = \frac{1}{6}\begin{pmatrix} 8 & -3 \\ 2 & 0 \\ -4 & 3 \end{pmatrix}$
(c) $\mathbf{AC} = \mathbf{I} \Rightarrow \mathbf{A}(\mathbf{Cb}) = \mathbf{b}$. $x_1 = 5/6$, $x_2 = 1/3$, $x_3 = -1/6$ (d) $r(\mathbf{A}\mathbf{A}') = m$ implies that $|\mathbf{A}\mathbf{A}'| \neq 0$, so $\mathbf{A}\mathbf{A}'$ has an inverse.

1-15. (a) $r(\mathbf{A}_a) = 3$ if $a \neq 0$ and $a \neq 1$. If $a = 0$ or $a = 1$, $r(\mathbf{A}_a) = 2$.
(b) $\lambda_1 = 0,\ \mathbf{x}_1 = \alpha\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $\lambda_2 = -1,\ \mathbf{x}_2 = \beta\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; $\lambda_3 = 2,\ \mathbf{x}_3 = \gamma\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$
(c) If $a \neq 0$, $a \neq 1$, $b \neq 0$, and $b \neq 1$, then $r(\mathbf{A}_a\mathbf{A}_b) = 3$. Otherwise the rank is 2.

1-16. (a) $\lambda = 3$ (multiplicity 3), $\lambda = 7$ (multiplicity 1)
(b) $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ for $\lambda = 3$; $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda = 7$.

1-17. (a) $\lambda_1 = 3$ with $\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\lambda_2 = \lambda_3 = 0$, with $\mathbf{w} = r\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
(b) $p = a - b$, $q = b$ (c) Easy verification. (d) $\mu_1 = a + 2b$, $\mu_2 = \mu_3 = a - b$.

1-18. (a) $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 1$ (b) No, $\mathbf{B}\mathbf{x}_1 = 4\mathbf{x}_1$ (c) $\mathbf{C} + \mathbf{I}_n$ has an inverse iff -1 is not an eigenvalue of $\mathbf{C}$. But $\lambda = -1$ does not satisfy $\lambda^3 = \lambda^2 + \lambda$, so $\mathbf{C} + \mathbf{I}_n$ has an inverse.

1-19. (a) $\lambda_1 = 0$, $\lambda_2 = c$, $\lambda_3 = (1 - a^2)b$. (b) Direct verification. (c) $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_2 = 4$. (d) $\mathbf{P} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$, $\mathbf{C} = \dfrac{1}{1 - a^2}\begin{pmatrix} -2a^2 & 0 & 2a \\ 0 & 1 - a^2 & 0 \\ -2a & 0 & 2 \end{pmatrix}$

2-01. (a) (i) Strictly concave since $0.5 + 0.3 < 1$. (ii) Quasiconcave.
(b) The Hessian matrix is $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -4 & 0 \\ 1 & 0 & -10 \end{pmatrix}$. The leading principal minors are $D_1 = -2$, $D_2 = 7$ and $D_3 = -66$. So F is strictly concave. (c) G is defined provided $\ln(x + y - 4)$ is defined and greater than or equal to 0, i.e. if $x + y - 4 \geq 1$, and so $x + y \geq 5$. Here $\ln(x + y - 4)$ is an increasing concave function of a concave function, so concave. Since $u \mapsto \sqrt{u}$ is increasing and concave, G is concave.
(d) The Hessian matrix is $\begin{pmatrix} 6 & -2 & -4 \\ -2 & 2 & 0 \\ -4 & 0 & 4 \end{pmatrix}$. The determinant is 0, so H is not strictly concave/convex. The 3 principal minors of order 2 are all equal to 8. The three principal minors of order 1 (the diagonal elements) are all positive. We conclude that H is convex. (In fact, $H = (x_1 - x_2)^2 + 2(x_1 - x_3)^2 \geq 0$ everywhere, but for (say) $x_1 = x_2 = x_3 = 1$ we have $H = 0$. So H is not positive definite.)

2-02. The Hessian matrix is $\begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix} = \begin{pmatrix} a^2 e^{ax+by^2} & 2abye^{ax+by^2} \\ 2abye^{ax+by^2} & 2b(1 + 2by^2)e^{ax+by^2} \end{pmatrix}$. We

4145ex3 29.4.2008 621

see that $\Delta = f''_{11}f''_{22} - (f''_{12})^2 = 2a^2be^{2ax+2by^2}$. For $b \geq 0$, $f''_{11} \geq 0$, $f''_{22} \geq 0$ and $\Delta = f''_{11}f''_{22} - (f''_{12})^2 \geq 0$, so f is convex. If $b < 0$ and $a \neq 0$, then $\Delta < 0$, so f is neither convex nor concave. If $b < 0$ and $a = 0$, $f(x,y) = e^{by^2}$, which is quasiconcave, but neither convex nor concave.

For $b \geq 0$ we can also argue in this way: $u = ax + by^2$ is then convex, and $u \mapsto e^u$ is increasing and convex, so $f(x,y)$ is convex.

2-03. $f(x,y)$ is concave as a sum of concave functions. (Alternatively: Look at the Hessian.)

2-04. The Hessian is $\mathbf{H} = \begin{pmatrix} e^{-x}\sqrt{1+y^2} & -ye^{-x}/\sqrt{1+y^2} \\ -ye^{-x}/\sqrt{1+y^2} & e^{-x}/(1+y^2)^{3/2} \end{pmatrix}$, and we see that $|\mathbf{H}| = e^{-2x}(1-y^2)/(1+y^2)$. The conclusion follows.

2-05. The Hessian is $\mathbf{H} = \begin{pmatrix} -12\sin 2x & 0 \\ 0 & -6 \end{pmatrix}$. Then $|\mathbf{H}| = 72\sin 2x$. The conclusion follows since $2x \in (0,\pi)$.

2-06. (a) The Hessian matrix is here $\begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix} = \begin{pmatrix} (x-y)^2-4 & -(x-y)^2 \\ -(x-y)^2 & (x-y)^2-4 \end{pmatrix}$. Thus $\Delta = f''_{11}f''_{22} - (f''_{12})^2 = 8(2-(x-y)^2)$. We see that if (say) x is close to -1 and y is close to 1, then $\Delta < 0$, so f is neither convex nor concave in D.
(b) The function is concave in that part of D which is between the lines $x-y = -\sqrt{2}$ and $x - y = \sqrt{2}$. Firstly, $f''_{11} = f''_{22} = (x-y)^2 - 4 \leq 0$ for all (x,y) because $(x-y)^2$ is obviously less than 4 in D. Secondly, $\Delta \geq 0$ iff $2-(x-y)^2 \geq 0$ iff $(x-y)^2 \leq 2$ iff $-\sqrt{2} \leq x - y \leq \sqrt{2}$.

2-07. $\mathbf{H} = \begin{pmatrix} a(\ln x)^{a-2}(\ln y)^b(a-1-\ln x)/x^2 & ab(\ln x)^{a-1}(\ln y)^{b-1}/xy \\ ab(\ln x)^{a-1}(\ln y)^{b-1}/xy & b(\ln x)^a(\ln y)^{b-2}(b-1-\ln y)/y^2 \end{pmatrix}$, and $|\mathbf{H}| = abx^{-2}y^{-2}(\ln x)^{2a-2}(\ln y)^{2b-2}[1-(a+b)+(1-a)\ln y + (1-b)\ln x + \ln x \ln y]$. The conclusion follows.

3-01. $x = 5$, $y = 2.5$

3-02. (a) The Kuhn–Tucker conditions: There must exist a λ such that $2x - 10\lambda x - 6\lambda y = 0$, $2y - 6\lambda x - 10\lambda y = 0$, $\lambda \geq 0$, and $\lambda = 0$ if $5x^2 + 6xy + 5y^2 < 1$. The points that satisfy these conditions are: (i) $(0,0)$ with $\lambda = 0$; (ii) $(1/4, 1/4)$ and $(-1/4,-1/4)$ with $\lambda = 1/8$; (iii) $(1/2,-1/2)$ and $(-1/2,1/2)$ with $\lambda = 1/2$. The points in (iii) solve the problem. (b) $f(x,y)$ measures the square of the distance from (x,y) to $(0,0)$. (c) $\Delta f_{\max} \approx \frac{1}{2} \cdot 0.1 = 0.05$, $\Delta f_{\min} \approx \frac{1}{8} \cdot 0.1 = 0.0125$

3-03. (a) (i) $1 + y - 2\lambda xe^y = 0$ (ii) $x - \lambda - \lambda x^2 e^y = 0$ (iii) $\lambda \geq 0$ ($\lambda = 0$ if $y + x^2e^y < 1$)
(b) $(x,y) = (0,-1)$ with $\lambda = 0$, $(x,y) = (1,0)$ with $\lambda = 1/2$.

3-04. $x = 0$, $y = 0$, $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 0$. ($\mathcal{L} = -e^{x+y} - e^y - 2x - y - \lambda_1(-x-y) - \lambda_2(-x) - \lambda_3(-y)$. Look at the eight possibilities: All λ's are 0, any two of them are 0, any one of them is 0, none of them are 0.)

3-05. There exist numbers $\lambda \geq 0$, $\mu \geq 0$ and $\sigma \geq 0$ such that

$$\begin{aligned} 2xye^{-x-y} - x^2ye^{-x-y} + \lambda + \sigma &= 0 \\ x^2e^{-x-y} - x^2ye^{-x-y} + \mu + \sigma &= 0 \\ \lambda(x-1) &= 0 \\ \mu(y-1) &= 0 \\ \sigma(x+y-4) &= 0 \end{aligned}$$

(b) The only solution: $(x,y) = (8/3, 4/3)$ (c) Yes

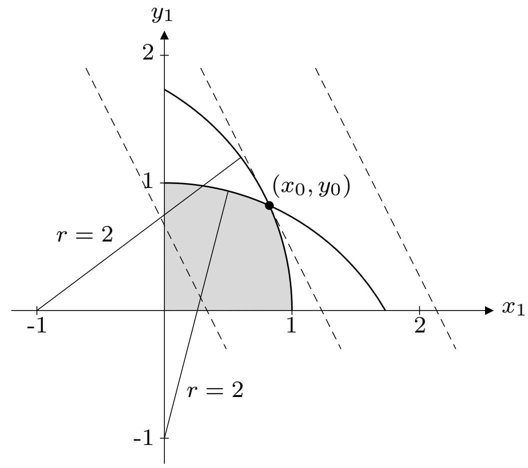

Figure 3-07

3-06. (a) $(x, y, z) = (\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}, 1)$, $\lambda = \frac{1}{4}\sqrt{2}$, $\mu = \frac{1}{2}$
(b) The maximum value of f increases by approximately 0.01.

3-07. (a) See Figure 3-07.The curved lines are arcs of circles around $(-1, 0)$ and $(0, -1)$.
(b) $x_0 = y_0 = \frac{1}{2}(\sqrt{7} - 1)$.
(c) The Kuhn–Tucker-conditions: There exist numbers $\lambda \geq 0$ and $\mu \geq 0$ such that

$$\begin{aligned} 2 - \lambda 2(x_0 + 1) - \mu 2x_0 &\leq 0 \\ 1 - \lambda 2y_0 - \mu 2(y_0 + 1) &\leq 0 \\ x_0(2 - \lambda 2(x_0 + 1) - \mu 2x_0) &= 0 \\ y_0(1 - \lambda 2y_0 - \mu 2(y_0 + 1)) &= 0 \\ \lambda((x_0 + 1)^2 + y_0^2 - 4) &= 0 \\ \mu(x_0^2 + (y_0 + 1)^2 - 4) &= 0 \end{aligned}$$

(d) The change $\approx \mu \cdot 0.1 = \dfrac{3 - \sqrt{7}}{4\sqrt{7}} \cdot 0.1 \approx 0.003$.

3-08. $(x, y) = (\sqrt[3]{2}, (\sqrt[3]{2}\,)^2)$ with $\lambda_1 = 1 - \dfrac{2}{\left(\sqrt[3]{2}\right)^2 + \left(\sqrt[3]{2}\right)^5}$, $\lambda_2 = \lambda_3 = 0$

3-09. (a) $(x, y) = (-\frac{1}{5}\sqrt{15}, -\frac{1}{5}\sqrt{15}\,)$ (b) $f(x) = x^5 - x^3$, $x \leq 1$. Max. at $x = -\frac{1}{5}\sqrt{15}$.

3-10. (a) $c \geq \frac{3}{2}$ (b) (i) $\partial L/\partial x = -4x^3 - 8x + 6y + a - \lambda = 0$
(ii) $\partial L/\partial y = -4y^3 + 6x - 12y + b - 2\lambda y + \mu = 0$ (iii) $\lambda \geq 0$ ($= 0$ if $x + y^2 < 1$)
(iv) $\mu \geq 0$ ($= 0$ if $y > -1$) (c) $a \geq 6$, $2a + b \leq -4$

3-11. (b) (i) $\partial L/\partial x = a - 12x - 5y - 4x(x^2 + y^2) + \lambda + \dfrac{1}{2\sqrt{x}}\,\mu = 0$
(ii) $\partial L/\partial y = b - 5x - 10y - 4y(x^2 + y^2) - \mu = 0$ (iii) $\lambda \geq 0$ ($\lambda = 0$ if $1 - x < 0$)
(iv) $\mu \geq 0$ ($\mu = 0$ if $y - \sqrt{x} < 0$) (c) $a \leq 16$, $b = 5$ (d) $b = 512$

3-12. (a) $(x^*, y^*) = (\ln(3/2), 2/3)$ (b) See Figure 3-12. The problem is to find the minimal distance from $(-\frac{1}{2}, 0)$ to the admissible set.

3-13. (a) (i) $\partial L/\partial x = -2 + \lambda \leq 0$ ($= 0$ if $x > 0$) (ii) $\partial L/\partial y = -1 + \lambda \leq 0$ ($= 0$ if $y > 0$)
(iii) $\partial L/\partial z = \dfrac{a}{z + 1} - 1 - 2\lambda z \leq 0$ ($= 0$ if $z > 0$) (iv) $\lambda \geq 0$ ($\lambda = 0$ if $z^2 < x + y$)
(b) For $a \leq 1$, $x = y = z = 0$ satisfy all the conditions if $0 \leq \lambda \leq 1$. For $a > 1$, $x = 0$, $y = z^2$ and $z = \frac{1}{4}(-3 + \sqrt{1 + 8a}\,)$ satisfy all the conditions with $\lambda = 1$.

3-14. (a) Minimum at $(x^*, y^*) = (1, 1)$. (b) Find the point in the admissible domain with the minimal distance to $(2, 2)$.

4145ex3 29.4.2008 621

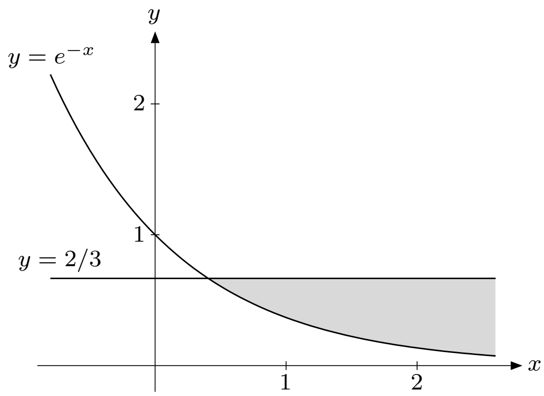

Figure 3-12

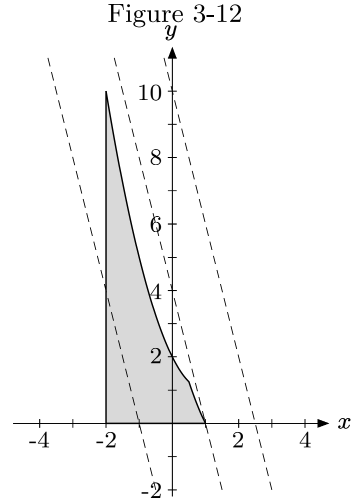

Figure 3-15

3-15. (a) See Figure 3-15. (b) Maximum at $(x, y) = (1, 0)$
(c) With $L = 4x + y - \lambda_1(y - (x-1)^2) - \lambda_2(y - (x-2)^2) - \lambda_3(-x) - \lambda_4(-y) - \lambda_5 x$, the necessary conditions are :
(i) $L'_1 = 4 + 2\lambda_1(x-1) + 2\lambda_2(x-2) + \lambda_3 - \lambda_5 = 0$ (ii) $L'_2 = 1 - \lambda_1 - \lambda_2 + \lambda_4 = 0$
(iii) $\lambda_1 \geq 0$ ($\lambda_1 = 0$ if $y - (x-1)^2 < 1$) (iv) $\lambda_2 \geq 0$ ($\lambda_2 = 0$ if $y - (x-2)^2 < -1$)
(v) $\lambda_3 \geq 0$ ($\lambda_3 = 0$ if $x > -2$) (vi) $\lambda_4 \geq 0$ ($\lambda_4 = 0$ if $y > 0$) (vii) $\lambda_5 \geq 0$ ($\lambda_5 = 0$ if $x < 2$)

3-16. $(x^*, y^*, z^*) = (1, 1, 2)$

3-17. $(x^*, y^*, z^*) = (2/3,\ 2/3,\ -1/3)$, with Lagrange multipliers $\lambda = 3/26$ and $\mu = 1/13$.

3-18. $(x^*, y^*) = (1, 1)$

3-19. $(x, y, z) = (0, 0, 1)$, $\lambda = 0$, $\mu = e$

3-20. (a) With the Lagrangian $\mathcal{L} = x + y - 1 - \lambda(x + y + z^2) - \mu(3x^2 + 2y + \frac{1}{3}z)$, the necessary conditions are:

$$\mathcal{L}'_x = 1 - \lambda - 6x\mu = 0 \qquad (1)$$
$$\mathcal{L}'_y = 1 - \lambda - 2\mu = 0 \qquad (2)$$
$$\mathcal{L}'_z = -2\lambda z - \tfrac{1}{3}\mu = 0 \qquad (3)$$
$$\lambda \geq 0 \quad (\lambda = 0 \text{ if } x + y + z^2 < a) \qquad (4)$$
$$\mu \geq 0 \quad (\mu = 0 \text{ if } 3x^2 + 2y + \tfrac{1}{3}z < 0) \qquad (5)$$

(b) For $a = 0$: $0 \leq x \leq 2/3$, $y = -x$, $z = 0$, $\lambda = 1$, $\mu = 0$.
(c) For $a = 1$: $x = 1/3$, $y = -1/36$, $z = -5/6$, $\lambda = 1/11$, $\mu = 5/11$.

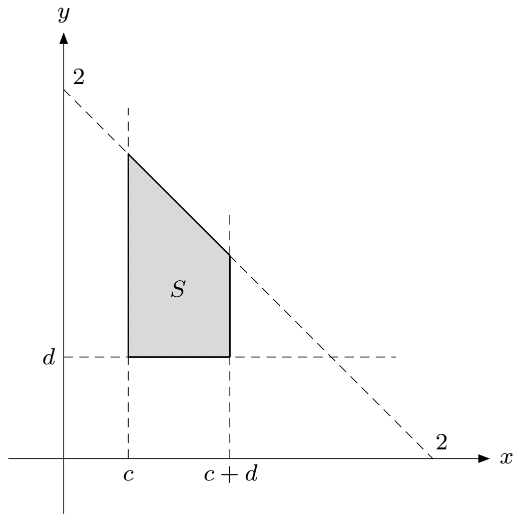

Figure 3-21

3-21. (a) For $x = c$ or $y = d$, $f(x, y) = 0$. For x slightly greater than c and y slightly greater than d, (x, y) still lies in S and $f(x, y) > 0$. Therefore (1) and (2) cannot be binding at the optimum. The set S of admissible points is shown in Figure 3-21.
(b) Note that $\alpha \ln(x - c) + (1 - \alpha)\ln(y - d) = \ln f(x, y)$.
(c) If $\alpha(2 - d) \le d + \alpha c$, then $x^* = \alpha(2 - d) + c(1 - \alpha)$ and $y^* = 2 - x^*$.
If $\alpha(2 - d) > d + \alpha c$, then $x^* = c + d$ and $y^* = 2 - c - d$.

3-22. (a) (i) $\partial L/\partial x = -2x + \lambda y - 2\mu x = 0$ (ii) $\partial L/\partial y = -2y + \lambda x - 2\mu y = 0$
(iii) $\partial L/\partial z = 4 - 2z - \lambda - 2\mu z = 0$ (iv) $\lambda \ge 0$ ($\lambda = 0$ if $z < xy$)
(v) $\mu \ge 0$ ($\mu = 0$ if $x^2 + y^2 + z^2 < 3$) (vi) $z \le xy$ (vii) $x^2 + y^2 + z^2 \le 3$
(b) $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (-1, -1, 1)$ both solve the problem, with $\lambda = 2$, $\mu = 0$. (c) $\lambda \cdot \Delta b_1 = 2 \cdot 0.1 = 0.2$

3-23. (a) $(x^*, y^*, z^*) = \big(2\sqrt{6}/3\,,\, 2\sqrt{6}/3\,,\, 2\sqrt{6}/3\big)$ with $\lambda = 0$ and $\mu = \sqrt{6}/3$.
(b) If we let $y = 1$ and $z = -1$, then $(x, y, z) = (x, 1, -1)$ is an admissible point for all $x \le 5$, and $xyz = -x$ can be made arbitrary large by choosing x a sufficiently large negative number.

4-01. $F'(T) = f(T) + r\displaystyle\int_0^T f(t)e^{-r(t-T)}\,dt = f(T) + rF(T)$.

4-02. (a) $F'(x) = e^{x^3} + \displaystyle\int_0^x t^2 e^{xt^2}\,dt$, $F''(x) = 4x^2 e^{x^3} + \int_0^x t^4 e^{xt^2}\,dt$
(b) $F''(x)$ is ≥ 0 on $(0, \infty)$.

4-03. $g'(x) = \displaystyle\int_\pi^{2\pi} \cos(xt)\,dt$, $g'(1) = \displaystyle\int_\pi^{2\pi} \cos t\,dt = 0$.

4-04. $\dot{y} = \sin^2 2t + \displaystyle\int_0^t 2\sin(t + x)\cos(t + x)\,dx$. Differentiate once more w.r.t. t.

4-05. $\frac{1}{2}(e - 1)$

4-06. $140/9$

4-07. $\frac{4}{15}(2\sqrt{2} - 1)$

4-08. $1/2$. (*Hint:* Change the order of integration.)

4-09. $\dfrac{1}{2\pi^2}$. (The innermost integral is $\displaystyle\int_0^\pi \frac{x}{y^3}\cos\Big(\frac{x^2}{y}\Big)dx = \frac{1}{2y^2}\sin\frac{\pi^2}{y}$.)

4-10. $V'(t) = \int_1^t F(t,x,t)\,dx + \int_1^t F(t,t,y)\,dy + \int_1^t \Big(\int_1^t \frac{\partial F(t,x,y)}{\partial t}\,dx\Big)\,dy$

4-11. $\frac{dT(t)}{dt} = \frac{z(t)}{x(t+T(t))} - 1, \quad \frac{dT(t)}{dt} = \frac{\bar{y}(t)}{\bar{y}(t+T(t))} - 1$

5-01. (a) $x = \frac{1}{5}(2e^{-4t} + 3e^t)$
(b) $x = \frac{1}{3}\ln\big(6\sqrt{t+8} - 18\ln(3+\sqrt{t+8}\,) + C\big)$, $C = 18\ln 6 - 17$.

5-02. The solution is given implicitly by $\frac{3}{5a^2}(ax+b)^{5/3} - \frac{3b}{2a^2}(ax+b)^{2/3} = \frac{1}{3}t^3 + C$, where C is a constant. The left-hand side of this equation can also be written as $\frac{3x}{2a}(ax+b)^{2/3} - \frac{9}{10a^2}(ax+b)^{5/3}$ or $\frac{3}{10a^2}(ax+b)^{2/3}(2ax-3b)$. The differential equation also has the constant solution $x \equiv -b/a$ (provided $b \neq 0$).

5-03. $x(t) = 1 - 2/(t^2-1)$

5-04. (a) $x(t) = 2e^{-2t} - e^{-4t}$ (b) $k(t) = \Big(k_0 + \frac{a}{\alpha}\Big)e^{(2\alpha s/3)[(t+1)^{3/2}-1]} - \frac{a}{\alpha}$

5-05. (a) $x = Ce^{-2t} + 1$. (b) $w(t) = -e^{-2t} + t + 1$.

5-06. (a) $\dot{K} = \gamma K^\alpha(\beta t + L_0)$ (b) $K = \Big((1-\alpha)\gamma\big(\frac{\beta}{2}t^2 + L_0 t\big) + K_0^{1-\alpha}\Big)^{1/(1-\alpha)}$

5-07. $x = Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$

5-08. $x = 5\sqrt{3e^{-t}+1}$, $\dot{x}(0) = -15/4$

5-09. $x = \Big(\frac{4}{\varphi(t)^2} - 9\Big)^{1/3}$, where $\varphi(t) = t\ln t - t + 2/3$.

5-10. $x = \frac{1}{2}\big[3t^2\cos(t^2) - 3\sin(t^2) + 1\big]^{-1/3} - \frac{1}{2}$.

5-11. $x = e^{t^2-2t}(3 - 2\sqrt{2} + 2\sqrt{1+e^t}\,)$

5-12. $x(t) = -\frac{1}{2}e^{1-t^3} + \frac{1}{2}e^{t^2-t^3}$

5-13. (a) $x = C(t+1)^2$ (b) Differential equation for u: $\dot{u} = 2(t+1)^{-2} + (t+1)^2$. The general solution of $(*)$: $x = C(t+1)^2 - 2(t+1) + \frac{1}{3}(t+1)^5$.

5-14. $x = -1 \pm (Ae^{6t} + \frac{1}{3})^{-1/2}$ or $x \equiv -1$.

6-01. (a) $x = C_1e^{8t} + C_2 + 17t/8$ (b) $x = e^{-t}(A\sin 2t + B\cos 2t)$

6-02. (a) $x = C_1e^{-4t} + C_2e^{\frac{1}{2}t}$ (b) $x = C_1e^{-4t} + C_2e^{\frac{1}{2}t} - \frac{1}{2}t - \frac{7}{8} - \frac{12}{85}\sin t - \frac{14}{85}\cos t$

6-03. (i) $x = C_1e^{2t} + C_2e^{7t/4}$ (ii) $x = C_1e^{2t} + C_2e^{7t/4} + \frac{1}{14}t + \frac{15}{196} + \frac{2}{65}\sin t + \frac{3}{65}\cos t$

6-04. $x(t) = e^{3t}(A\cos 4t + B\sin 4t) + \frac{1}{25}t + \frac{6}{625}$

6-05. $x = C_1e^{-3t} + C_2e^{-t/3} + u^*(t)$ with (i) $u^*(t) = 0$, (ii) $u^*(t) = -2e^{-t} + 2$, (iii) $u^*(t) = te^{-3t}$.

6-06. $x = e^{-t/2}\Big(C_1\cos\Big(\frac{\sqrt{7}}{2}t\Big) + C_2\sin\Big(\frac{\sqrt{7}}{2}t\Big)\Big) + \frac{t^2}{2} - \frac{t}{2} + \frac{3}{4}$

6-07. (i) $x = Ae^{r_1t} + Be^{r_2t}$, $r_{1,2} = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - a + 2}$
(ii) $x = Ae^{r_1t} + Be^{r_2t} + \frac{t}{a-2} + \frac{a}{(a-2)^2}$

6-08. (i) $g(t) = Ae^{-t/4} + B$ (ii) $g(t) = C/(t+1) + D$

6-09. (a) $Y^* = -\frac{1}{l+m}t$ is a particular solution. (b) $Y = e^{-t/2}(A\cos t + B\sin t) - \frac{3}{20}t$.
(c) Oscillations if $\frac{1}{4}(\alpha l + \beta)^2 < \alpha\beta(l+m)$. For every solution Y we then have $Y(t) + \frac{1}{l+m}t \to 0$ as $t \to \infty$, that is, Y will approach the linear function Y^*.

4145ex3 29.4.2008 621

6-10. (a) $V(x) = Ax^{a_1} + Bx^{a_2}$, where $a_{1,2} = -\Big(\dfrac{\mu}{\sigma^2} - \dfrac{1}{2}\Big) \pm \sqrt{\Big(\dfrac{\mu}{\sigma^2} - \dfrac{1}{2}\Big)^2 + \dfrac{2\rho}{\sigma^2}}$.
(b) $V^*(x) = \dfrac{1}{\rho - \mu}x - \dfrac{w}{\rho}$ (c) $V(x) = Ax^2 + \dfrac{B}{x^2} + \dfrac{1}{3}x - \dfrac{5}{2}$

6-11. (a) $x = Ae^{r_1 t} + Be^{r_2 t}$ with $r_{1,2} = k - 1 \pm \sqrt{5 - 2k}$ for $k < \frac{5}{2}$,
$x = Ae^{(3/2)t} + Bte^{(3/2)t}$ for $k = \frac{5}{2}$,
$x = e^{(k-1)t}\big(A\cos\sqrt{2k-5}\,t + B\sin\sqrt{2k-5}\,t\big)$ for $k > \frac{5}{2}$.
(b) For $k \neq 2$, $k \neq 5/2$: $u^* = \dfrac{1}{2k^2 - 9k + 10}e^{(4-k)t}$. For $k = 2$: $u^* = te^{2t}$.
For $k = 5/2$: $u^* = t^2 e^{3t/2}$.
(c) $x = e^{2t}(\sin t - \cos t) + e^t$. $t = 0$ is a local minimum point.

6-12. (a) Use Leibniz's rule. (b) $p^* = \dfrac{a+c}{b+d}$. If $D = \frac{1}{4}\alpha^2 - \beta(b+d) > 0$ and $r_{1,2} = -\frac{1}{2}\alpha \pm \sqrt{D}$, then $p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + p^*$. If $D = 0$, $p(t) = (C_1 + C_2 t)e^{-\alpha t/2} + p^*$.
If $D < 0$ and $\gamma = \sqrt{-D}$, $p(t) = e^{-\alpha t/2}(C_1 \cos\gamma t + C_2 \sin\gamma t) + p^*$.
(c) Oscillations if and only if $D = \frac{1}{4}\alpha^2 - \beta(b+d) < 0$.

6-13. (a) $x = \dfrac{1}{2}At + \dfrac{B}{t}$ (b) $x(t) = t$

6-14. If $k \neq -5$, $x = Ae^{-5t} + Bte^{-5t} + \frac{5}{(k+5)^2}e^{kt} + \frac{6}{169}\sin t - \frac{5}{338}\cos t$.
For $k = -5$, $x = Ae^{-5t} + Bte^{-5t} + \frac{5}{2}t^2 e^{-5t} + \frac{6}{169}\sin t - \frac{5}{338}\cos t$.

6-15. (a) $x = e^{-2t}(A\cos 3t + B\sin 3t) + \frac{1}{50}e^{2t}$.
(b) Look for a particular integral of the form $C\sin 3t + D\cos 3t$.

7-01. $\ddot{x} - 3\dot{x} + 3x = 1 - 2t$. $x = e^{3t/2}\Big(C_1 \cos\dfrac{\sqrt{3}}{2}t + C_2 \sin\dfrac{\sqrt{3}}{2}t\Big) - \dfrac{2t}{3} - \dfrac{1}{3}$,
$y = e^{3t/2}\Big(\dfrac{C_1 + C_2\sqrt{3}}{2}\cos\dfrac{\sqrt{3}}{2}t + \dfrac{C_2 - C_1\sqrt{3}}{2}\sin\dfrac{\sqrt{3}}{2}t\Big) - \dfrac{t}{3} - \dfrac{1}{3}$.

7-02. (a) For $a \neq \pm 2$,
$x = Ae^{(a-2)t} + Be^{(a+2)t} + \dfrac{2\beta - a\alpha}{a^2 - 4}$, $y = -Ae^{(a-2)t} + Be^{(a+2)t} + \dfrac{2\alpha - a\beta}{a^2 - 4}$.
For $a = 2$, $x = A + Be^{4t} + \frac{1}{2}(\alpha - \beta)t$, $y = -A + Be^{4t} - \frac{1}{2}(\alpha - \beta)t - \frac{1}{4}(\alpha + \beta)$.
For $a = -2$, $x = Ae^{-4t} + B + \frac{1}{2}(\alpha + \beta)t$, $y = -Ae^{-4t} + B + \frac{1}{2}(\alpha + \beta)t + \frac{1}{4}(\beta - \alpha)$.
(b) $(x^*, y^*) = \Big(\dfrac{2\beta - a\alpha}{a^2 - 4}, \dfrac{2\alpha - a\beta}{a^2 - 4}\Big)$. If $a < -2$, (x^*, y^*) is (globally) asymptotically stable. If $-2 < a < 2$, (x^*, y^*) is a saddle point. (c) $x = e^{-3t} + 2$, $y = -e^{-3t} + 3$

7-03. (a) $(-1, 0)$: no decision; $(0, 0)$: no decision; $(1, -2)$: saddle point.
(b) $x(t) = 1/(t+1)$, $y(t) = -1/(t+1)$

7-04. (a) $(0, 0)$ is a saddle point; $(1/2, 1/2)$ is locally asymptotically stable.
(b) $\dot{z} = z - z^2$; $z = Ae^t/(1 + Ae^t)$ or $z \equiv 1$. (c) (i) $x(t) = y(t) = e^t/(2e^t - 1)$,
(ii) $x(t) = y(t) = e^t/(2 + 2e^t)$, (iii) $x(t) = y(t) = e^t/(2e^t - 3)$.

7-05. (a) $(0, 0)$ is a locally asymptotically stable equilibrium point. $(2, 4)$ and $(-2, -4)$ are saddle points. (b) See Figure 7-05(a) and (b). (c) $\frac{1}{2}(7 + \sqrt{41}) \approx 6.702$.

7-06. (a) See Figure 7-06. (b) The only equilibrium point is $(0, 0)$. Not stable.
(c) $x(t) = -e^{-t}$, $y(t) = (z(t))^{-1}$, where $z(t) = e^{e^{-t}}(e^{-1} + \int_0^t e^{-e^{-\tau}}\, d\tau)$.
$(x(t), y(t)) \to (0, 0)$ as $t \to \infty$.

8-01. (a) $\ddot{x} = -e^{-t}$ (b) $x = -e^{-t} + e^{-1} \cdot t + 1$

8-02. The Euler equation is $\ddot{x} - 4\dot{x} - 5x = -2e^{-t}$. Solution: $x^* = \frac{1}{3}(5 + t)e^{-t}$

4145ex3 29.4.2008 621

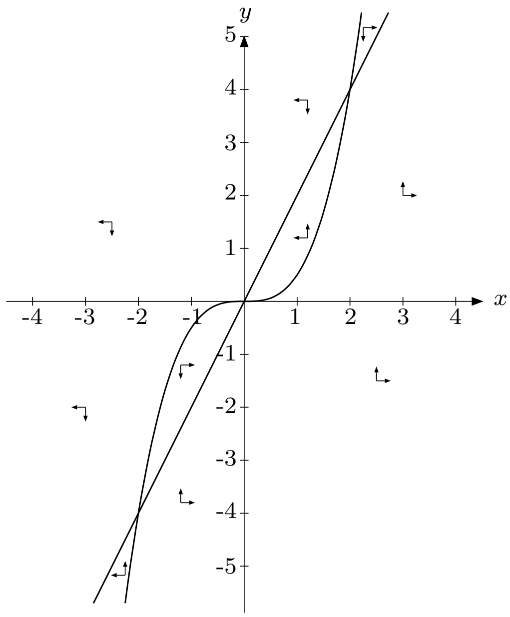

Figure 7-05(a). An “arrow diagram”.

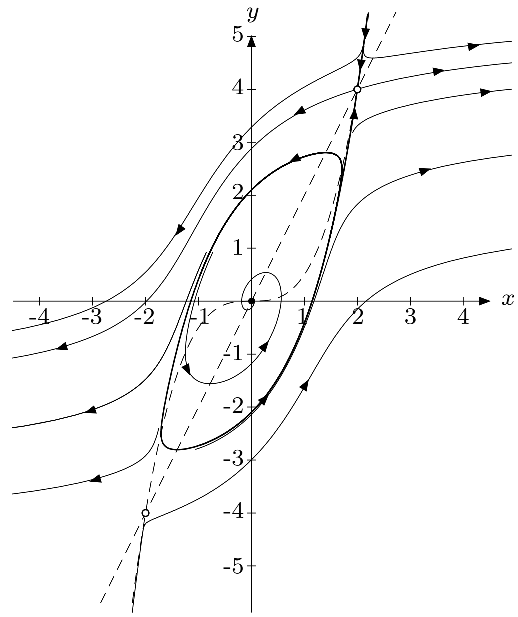

Figure 7-05(b). Some integral curves.

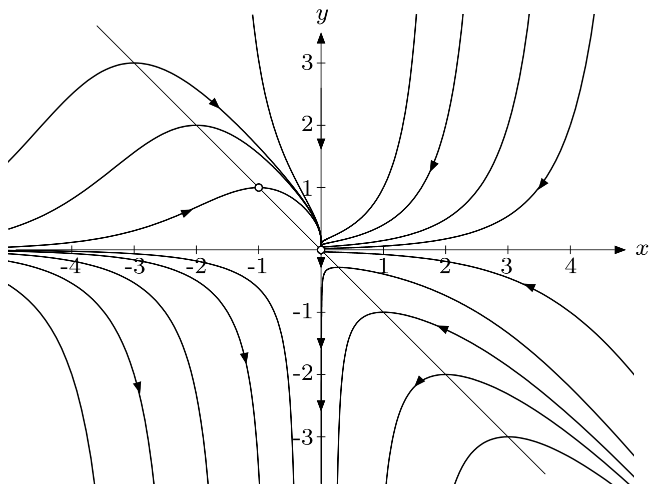

Figure 7-06.

8-03. (a) $\ddot{x} - \frac{1}{10}\dot{x} = -\frac{1}{200}t$. The general solution is $x = A + Be^{t/10} + \frac{1}{40}t^2 + \frac{1}{2}t$.
(b) $A = \dfrac{25}{2(1-e)}$, $B = -A$.

8-04. (a) $\ddot{K} - r\dot{K} - [(2b + rc)/2d]K = 0$ (b) $4bd \geq c^2$
(c) $K^*(t) = 4(e^{15} - e^{-5})^{-1}(e^{3t/2} - e^{-t/2})$

8-05. (a) $2t^2\ddot{x} + 4t\dot{x} - x = 0$ (b) $x^*(t) = \dfrac{t^{a_1} - t^{a_2}}{2^{a_1} - 2^{a_2}}$ with $a_{1,2} = \frac{1}{2}(-1 \pm \sqrt{3}\,)$.

8-06. (a) $\ddot{x} - \dfrac{pb^2 + qa^2}{q}x = 0$, $x = Ae^{\lambda t} + Be^{-\lambda t}$ if $\lambda = \sqrt{\dfrac{pb^2 + qa^2}{q}} > 0$, $x = At + b$ if $\lambda = 0$. (b) $x = \dfrac{e^t - e^{-t}}{e - e^{-1}}$.

8-07. (a) $a(t)\ddot{x} + \dot{a}(t)\dot{x} - \frac{1}{2}b(t) = 0$. (b) Put $y = \dot{x}$ and solve the linear differential equation in y. (c) $x(t) = \dfrac{5}{9\ln 3}\ln t + \dfrac{t^3 - 1}{18}$.

8-08. (a) The Euler equation for (1) is $\ddot{x} - \frac{1}{10}\dot{x} = \frac{1}{10}$. General solution $x(t) = Ae^{t/10} + B - t$.

Solution of the problem: $x = 1 - t$. (b) $\ddot{x} - (\beta - r)\dot{x} = -\frac{1}{2}a(\alpha - r)e^{(\alpha-\beta)t}$.

8-09. (a) The Euler equation is $\ddot{x} \equiv 0$, and the solution is $x^*(t) \equiv 1$. (c) Yes.

8-10. (a) $g''(\dot{x})\ddot{x} - rg'(\dot{x}) - c(t) = 0$ (b) $x = \left(B + \dfrac{T}{r}\right)\dfrac{e^{rt}-1}{e^{rT}-1} - \dfrac{t}{r}$

(c) $\dot{x}(t) \geq 0$ for t in $[0, T] \iff Br^2 \geq e^{rT} - 1 - rT$

9-01. $x^*(t) = -\frac{1}{2\alpha}e^{2-t} + \frac{5}{2\alpha} + 5e^t + \frac{1}{2\alpha}e^{2+t} - \frac{5}{2\alpha}e^t$, $p(t) = 2e^{2-t} - 2$, $u^*(t) = \frac{1}{2\alpha}(p(t) - 3)$

9-02. (a) Necessary conditions:
(1) $u = u^*(t)$ maximizes $H = -(x^*(t) - u + 2)^2 e^{-rt} + p(t)(u - \delta x^*(t))$ for u in $\mathbb{R}$,
(2) $\dot{p}(t) = -\dfrac{\partial H^*}{\partial x} = 2(x^*(t) - u^*(t) + 2)e^{-rt} + \delta p(t)$,
(3) $\dot{x}^*(t) = u^*(t) - \delta x^*(t)$, $x^*(0) = x_0$, $x^*(T) = x_T$.
(c) $u^*(t) = Be^{0.5t} - 2 - \frac{1}{18}Ae^{-0.4t}$, $x^*(t) = Be^{0.5t} - 4 - \frac{5}{9}Ae^{-0.4t}$,

$$p(t) = Ae^{-0.5t}, \quad \text{where } A = \frac{36(3 - e^5)}{5(e^5 - e^{-4})}, \quad B = \frac{4(3 - e^{-4})}{e^5 - e^{-4}}.$$

9-03. (a) $u^*(t) = \frac{1}{2}(e^{2-t} - 1)$, $x^*(t) = \frac{1}{4}(e^{2+t} - e^{2-t}) - \frac{1}{2}(e^t - 1)$, $p(t) = e^{2-t} - 1$

(b) $x^*(t) = \begin{cases} e^t - 1 & \text{if } 0 \leq t \leq t^*, \\ e^t - \frac{1}{4}e^{2-t} - \frac{9}{4}e^{t-2} + \frac{1}{2} & \text{if } t^* \leq t \leq 2, \end{cases}$

with $t^* = 2 - \ln 3$.

9-04. (a) $u^*(t) = \dfrac{225}{e^{3/4} - 1}e^{0.15t}$, $x^*(t) = \dfrac{1500}{e^{3/4} - 1}\left(e^{0.15t} - 1\right)$, $p(t) = \dfrac{1350}{e^{3/4} - 1}$.
(b) (i) $u = u^*(t)$ maximizes $\left[-C(u,t)e^{-rt} + p(t)u\right]$ for $u \geq 0$,
(ii) $\dot{p}(t) = -\partial H^*/\partial x = 0$, (iii) $p(5) \geq 0$ ($= 0$ if $x^*(5) > 1500$).
(c) $u^*(t) = 300$, $x^*(t) = 300t$, $p(t) = g'(300)$.

9-05. (a) $a \leq 0$, $c \leq 0$ and $ac - b^2 \geq 0$ (b) $\ddot{x} - r\dot{x} - [(a + br)/c]x = \frac{1}{2}(dr/c)t^2 - (d/c)t$

(c) $\dfrac{e^{3t} - e^{-3t}}{3(e^3 - e^{-3})} - \dfrac{t}{3}$

9-06. (a) (1) $u = u^*(t)$ maximizes $2(x^*(t))^2 - \frac{1}{2}u^2 + p(t)u$ for u in $(-\infty, \infty)$,
(2) $\dot{p}(t) = -4x^*(t)$, $p(T) = 0$.
(b) $u^*(t) = p(t) = 2(\cos 2t - \sin 2t)$, $x^*(t) = \sin 2t + \cos 2t$
(c) The maximum principle gives the same suggestion as in (a), but no optimal control exists. (For the suggested control the objective function tends to infinity as c tends to infinity.)

9-07. (a) (i) $u = u^*(t)$ maximizes $(p(t) - x^*(t))x^*(t)u$ for u in $[0, 1]$
(ii) $\dot{p}(t) = -2(1 - u^*(t))x^*(t) - p(t)u^*(t)$ (iii) $p(T) = 0$ (iv) $\dot{x}^*(t) = u^*(t)x^*(t)$
(c) For $t \in [0, t^*] = [0, T - \frac{1}{2}]$, we have $u^*(t) = 1$, $x^*(t) = e^t$, $p(t) = e^{2t^* - t}$.
For $t \in (t^*, T] = (T - \frac{1}{2}, T]$ we have $u^*(t) = 0$, $x^*(t) = e^{t^*}$, $p(t) = 2(T - t)e^{t^*}$.

9-08. (a) (1) $I = I^*(t)$ maximizes $-cI^2 + \lambda I$ for I in $\mathbb{R}$,
(2) $\dot{\lambda}(t) - r\lambda(t) = -a + 2bK^*(t) + \delta\lambda(t)$, (3) $\lambda(T) = 0$.
(b) $\ddot{K}^* - r\dot{K}^* - \big(\delta(\delta + r) + b/c\big)K^* = -a/2c$

(c) $K^*(t) = Ae^{-0.1t} + Be^{0.3t} + 20$, $A = \dfrac{5e - 80e^4}{4e^4 + 1}$, $B = -(A + 20) = \dfrac{5e + 20}{4e^4 + 1}$.

(d) We have the same differential equation for K as in part (c), and with the same initial condition, $K(0) = 0$, so once again $K^*(t) = Ce^{-0.1t} - (C + 20)e^{0.3t} + 20$ for a suitable constant C. However, the transversality condition is now $\lambda(10) = e^2$. (With $S(t, K) = Ke^{0.2t}$, we have $S_2'(t, K) = e^{0.2t}$ and so $S_2'(10, K(10)) = e^2$.) Hence, $\dot{K}^*(10) + 0.02K^*(10) = \frac{1}{20}$. This equation determines the value of C.

9-09. (a) The adjoint function is $p(t) = 1 - e^{t-T}$, and $u^*(t) = 1$ if $ap(t) > e^{2t}$, $u^*(t) = 0$ if $ap(t) < e^{2t}$.

4145ex3 29.4.2008 621

(b) $t \le \ln 2 \implies u^*(t) = 1$ and $x^*(t) = 10e^{-t} - 5e^{-2t}$,
$t > \ln 2 \implies u^*(t) = 0$ and $x^*(t) = (15/2)e^{-t}$.

(c) $u^*(t) \equiv 0,\ x^*(t) = 5e^{-t}$.

9-10. (b) $x(t) = 1 - \frac{1}{3}t$, $u = \frac{2}{3} - \frac{1}{3}t$

9-11. $u^*(t) = \begin{cases} 1 & \text{if } t \le 1, \\ 0 & \text{if } t > 1, \end{cases}$ $x^*(t) = \begin{cases} t+1 & \text{if } t \le 1, \\ 1 & \text{if } t > 1, \end{cases}$ $y^*(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{if } t > 1, \end{cases}$ $p_1(t) = 2 - t,\ p_2(t) = -1/2$.

9-12. $u = 0$, $x = 0$ for t in $[0, 1]$; $u = 1$, $x = t - 1$ for t in $(1, 2]$.

9-13. (a) (i) $u = u^*(t)$ maximizes $\big(x^*(t)\big)^2 - x^*(t) + p(t)u$ for u in $[0, 1]$,
(ii) $\dot{p}(t) = -\partial H^*/\partial x = -2x^*(t) + 1$, $p(T) = 0$.
(b) Solution: For $T < \frac{3}{2}$: $u^*(t) = 0$, $x^*(t) = 0$, $p(t) = t - T$.
For $T \ge \frac{3}{2}$: $u^*(t) = 1$, $x^*(t) = t$, $p(t) = -t^2 + t + T^2 - T$.

9-14. (a) For $b \ge a(e^T - 1)$: $u^*(t) = 0$, $x^*(t) = x_0e^t$ and $p(t) = a(e^{T-t} - 1)$.
For $b < a(e^T - 1)$: $u^*(t) = 2$, $x^*(t) = (x_0 + 2)e^t - 2$ and $p(t) = a(e^{T-t} - 1)$ in $[0, t^*]$, while $u^*(t) = 0$, $x^*(t) = (x_0 + 2)e^t - 2e^{t-t^*}$ and $p(t) = a(e^{T-t} - 1)$ in $(t^*, T]$, where $t^* = T - \ln(1 + b/a)$.

9-15. $u^*(t) = 1,\ x^*(t) = \dfrac{1}{\sqrt{1-2t}},\ p(t) = 12\sqrt{11}(1 - 2t)^{3/2} + 46t - 23$ if $t \in [0, 7/22]$,
$u^*(t) = 0,\ x^*(t) = \frac{1}{2}\sqrt{11},\ p(t) = 8 - 24t$ if $t \in (7/22, 1/3]$.

9-16. $u^*(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ -1 & \text{if } 1 < t \le 2, \end{cases}$ $x^*(t) = \begin{cases} t & \text{if } 0 \le t \le 1, \\ 2-t & \text{if } 1 < t \le 2. \end{cases}$

9-17. $u^*(t) = \dfrac{1}{C-t},\ x^*(t) = t - \dfrac{1}{2(C-t)} + \dfrac{1}{2C},\ p(t) = C - t,$ with $C = \frac{1}{2}(1 + \sqrt{3}\,)$
(b) $u^*(t) = \begin{cases} \frac{1}{A-t} & \text{if } t > A-1 \\ 1 & \text{if } t \le A-1 \end{cases}$, $x^*(t) = \begin{cases} t - \frac{1}{2(A-t)} + 1 - \frac{1}{2}A & \text{if } t > A-1 \\ \frac{1}{2}t & \text{if } t \le A-1 \end{cases}$,
with $A = \frac{1}{2}(5 - \sqrt{5}\,)$.

10-01. $x_t = (\sqrt{10}\,)^t(A\cos(\theta t) + B\sin(\theta t))$, with $\cos\theta = -3\sqrt{10}/10$.

10-02. $x(t) = 5^t(C_1\cos\theta t + C_2\sin\theta t) + \frac{1}{20}$ with $\cos\theta = \frac{3}{5}$.

10-03. $x_t = A(-3)^t + B\,2^t + \frac{1}{24}\,5^t - \frac{1}{4}t - \frac{3}{16}$

10-04. $x_t = 2^t - 2(-6)^t - t^2 - 2t - 2$.

10-05. (a) $x_t = A\,2^t + B\big(\frac{1}{2}\big)^t + 4\cdot 3^t$. If $x_0 = 0$ and $x_1 = 2$, then $A = -\frac{16}{3}$, $B = \frac{4}{3}$.
(b) $x_t = A\,\alpha^t + B\,\alpha^{-t} - \dfrac{\beta K}{\alpha\beta^2 - (1+\alpha^2)\beta + \alpha}\,\beta^t$

10-06. (a) For $c = 0$: $x_t = \frac{2}{5}(-2)^t + \frac{3}{5}3^t$. For $c = 1$: $x_t = \frac{7}{15}(-2)^t + \frac{7}{10}\,3^t - \frac{1}{6}$.
(b) We obtain the equation from part (a) with $c = 0$, $x_0 = 1$ and $x_1 = 1$, and $y_t = \frac{3}{5}(-(-2)^t + 3^t)$.

11-01. (a) $J_T(x) = -x^2$, with $u_T^*(x)$ undetermined. $J_{T-1}(x) = -x^2/2$ with $u_{T-1}^* = -x/2$. $J_{T-2}(x) = -x^2/3$ with $u_{T-2}^* = -x/3$.
(b) We claim that $J_{T-k} = -x^2/(k+1)$ with $u_{T-k}^* = -x/(k+1)$. This is true for $k = 1$. Suppose it is true for $k = s$. Then $J_{T-(s+1)}(x) = \max_{u\in\mathbb{R}}(-u^2 + J_{T-s}(x+u)) = \max_{u\in\mathbb{R}}(-u^2 - (x+u)^2/(k+1))$. The maximizer for the $g(u) = -u^2 - (x+u)^2/(k+1)$ is $u = -x/(s+2)$. (Note that g is concave.) It follows that $J_{T-(s+1)}(x) = -u^2 - (x+u)^2/(k+1) = -x^2/(s+2)$, which is the given formula for $t = s+1$. It follows by induction that the suggested formula is valid for all k.

11-02. (a) $J_T(x) = x$ with $u_T^*(x) = 0$, $J_{T-1}(x) = 2x$ with $u_{T-1}^*(x)$ undetermined, $J_{T-2}(x)= 3x+1$ with $u_{T-2}^*(x) = 1$, $J_{T-3}(x) = 4x+3$ with $u_{T-3}^*(x) = 1$, $J_{T-4}(x) = 5x+6$ with $u_{T-4}^*(x) = 1$
(b) $J_{T-k}(x) = (k+1)x + \frac{1}{2}k(k-1)$ with $u_{T-k}^*(x) = 1$ for $k \geq 2$.

11-03. (a) $J_T(x) = \ln x$ with $u_T^*(x)$ undetermined. $J_{T-1}(x) = 2\ln(x/2)$ with $u_{T-1}^* = x/2$. $J_{T-2}(x) = 3\ln(x/3)$ with $u_{T-2}^* = x/3$. (b) $u_{T-k}^* = x/(k+1)$

11-04. (a) $J_T(x) = 2x^2$ for $u_T^*(x) = 1$. $J_{T-1}(x) = 3x^2$ for $u_{T-1}^* = 0$. ($J_{T-1}(x) = \max_{u\in[0,1]}\{x^2(1+u) + J_T(x(1-u))\} = x^2 \max_{u\in[0,1]}(1+u+2(1-u)^2\}$. The function to be maximized is convex in u and has its maximum $3x^2$ at $u=0$.) $J_{T-2}(x) = 4x^2$ with $u_{T-2}^* = 0$.
(b) $x_t^* = x_0$ for all t and $u_t^* = 0$ for $t = 0, \ldots, T-1$, $u_T^* = 1$. The maximum value of the objective function is $J_0(x_0) = (T+2)x_0^2$.

11-05. (a) $f_0(s,x,u) = \sqrt{u}$ for $s < T$, $f_0(T,x,u) = -x$.
$J_T(x) = \max_{u\in(0,\infty)}(-x) = -x$, with $u_T^*(x)$ undetermined.
$J_{T-1}(x) = \max_{u\in(0,\infty)}\big\{\sqrt{u} + J_T\big(2(x+u)\big)\big\} = \max_{u\in(0,\infty)}\big\{\sqrt{u} - 2(x+u)\big\}$.
Put $g_{T-1}(u) = \sqrt{u} - 2x - 2u$. Then $g'_{T-1}(u) = 1/2\sqrt{u} - 2 = 0$ for $u = 1/16$, and $g''_{T-1}(u) = -1/4u\sqrt{u} < 0$ for $u > 0$, so $u = 1/16$ is the maximizer, and $J_{T-1}(x) = -2x + 1/2^3$ for $u_{T-1}^*(x) = 1/2^4$.
$J_{T-2}(x) = \max_{u\in(0,\infty)}\big\{\sqrt{u} + J_{T-1}\big(2(x+u)\big)\big\} = \max_{u\in(0,\infty)}\big\{\sqrt{u} - 4x - 4u + \frac{1}{2^3}\big\}$. It is easy to see that the function $g_{T-2}(u) = \sqrt{u} - 4x - 4u + 1/2^3$ attains its maximum for $u = 1/2^6$, and $J_{T-2}(x) = -2^2x + (2^2-1)/2^4$ for $u_{T-2}^* = 1/2^6$.
(b) $J_{T-k}(x) = -2^k x + (2^k - 1)/2^{k+2}$, by induction.

11-06. (a) $J_T(x) = \max_{u\in(0,1]}(x + \ln u) = x$ for $u_T^*(x) = 1$. $J_{T-1}(x) = \max_{u\in(0,1]}(x + \ln u + J_T(x-u)) = \max_{u\in(0,1]}(2x + \ln u - u)$. The maximum is attained at $u = 1$, so $J_{T-1}(x) = 2x - 1$ for $u_{T-1}^*(x) = 1$. Further, $J_{T-2}(x) = \max_{u\in(0,1]}(x + \ln u + J_{T-1}(x-u)) = \max_{u\in(0,1]}(3x + \ln u - 2u - 1)$. We see that $J_{T-2}(x) = 3x - 2 - \ln 2$ for $u_{T-1}^*(x) = 1/2$.
(b) Suppose the formula is valid for $t = s$. Then $J_{T-(s+1)}(x) = \max_{u\in(0,1]}(x + \ln u + J_{T-s}(x-u)) = \max_{u\in(0,1]}(x + \ln u + (s+1)(x-u) - s - \ln(1\cdot 2\cdots s)$. We see that the maximizer is $u = 1/(s+1)$, and it follows easily that $J_{T-(s+1)}(x) = (s+2)x - (s+1) - \ln(1\cdot 2\cdots s\cdot(s+1))$, which is the given formula for $t = s+1$.

11-07. (a) $J_T(x) = \sqrt{x}$ with $u_T^*(x)$ undetermined. $J_{T-1}(x) = \sqrt{5}\sqrt{x}$ with $u_{T-1}^* = 4/5$. $J_{T-2}(x) = 3\sqrt{x}$ with $u_{T-2}^* = 4/9$. (b) Inserting $J_t(x) = k_t\sqrt{x}$ into the fundamental equation and cancelling $\sqrt{x}$ yields $k_{t-1} = \max_{u\in[0,1]}\{2\sqrt{u} + k_t\sqrt{1-u}\}$. The optimal choice of u is $u = 4/(k_t^2+4)$, and it follows that $k_{t-1} = \sqrt{k_t^2+4}$. (Note that with $k_T = 1$, $k_{T-1} = \sqrt{5}$ and $k_{T-2} = 3$.)