University of Oslo / Department of Economics **English**

ECON4140/ECON4145 Mathematics 3

Monday December 12 2011, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

You are required to state reasons for all your answers.

Problem 1 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

(a) $\mathbf{v}_1 = (1, 1, 2)'$ is an eigenvector of $\mathbf{A}$. Find the corresponding eigenvalue λ_1.

(b) $\lambda_2 = 4$ is an eigenvalue of $\mathbf{A}$. Find a corresponding eigenvector $\mathbf{v}_2$.

(c) Find an eigenvalue λ_3 of $\mathbf{A}$ so that $\lambda_1 \neq \lambda_3 \neq \lambda_2$, and a corresponding eigenvector $\mathbf{v}_3$.

(d) Decide whether $\mathbf{A}$ is positive or negative definite or semidefinite, or neither.

Problem 2

(a) Find the general solution of

$$\ddot{x}(t) - 2\dot{x}(t) + x(t) = -\sin t \qquad (\star)$$

(b) Consider the system of differential equations

$$\dot{x}(t) = 2x(t) - y(t)$$
$$\dot{y}(t) = x(t) + \sin t$$

Show that x must solve equation $(\star)$, and use this to find the general solution of the system.

Problem 3

(a) The standard multivariate normal distribution has density function

$$f(x_1, \dots, x_n) = (2\pi)^{-n/2} e^{-(x_1^2+x_2^2+\cdots+x_n^2)/2}.$$

Show that f is quasiconcave. (Hint: Do not calculate determinants.)

In the following, let $a > 0$ be a constant, and consider the nonlinear programming problem

$$\max g(x, y) \qquad \text{subject to} \quad y \geq 0 \quad \text{and} \quad y \leq (x-a)^{2011} \tag{P}$$

where $g(x, y) = (2\pi)^{-1} e^{-(x^2+y^2)/2}$ (that is, as f but with $n = 2$).

(b) Show that there is no point that satisfies the associated Kuhn–Tucker conditions.

(c) Explain by a geometric argument why the problem (P) has a solution. Explain then how there yet can be no points satisfying the Kuhn–Tucker conditions.

Problem 4 Let b be a given continuous function which is strictly positive for all t, and define $B(t)$ as $B(t) = \int_0^t b(s)\, ds$. Consider the optimal control problem

$$\max \int_0^T b(t) \ln u(t)\, dt \qquad \text{where} \quad \dot{x} = rx - u, \quad x(0) = x_0, \quad x(T) \geq 0$$

where r, x_0 and T are positive constants, and u is allowed to take any positive value.

(a) Show that the pair (x^*, u^*) defined by

$$u^*(t) = x_0 e^{rt} \frac{b(t)}{B(T)}$$
$$x^*(t) = x_0 e^{rt} \left(1 - \frac{B(t)}{B(T)}\right)$$

satisfies all the necessary conditions from the maximum principle.

(b) Show that (x^*, u^*) solves the problem.