An infinite-horizon dynamic programming problem: Let $q \in (0,1)$ and $p \in \mathbb{R}$ be given constants, and consider the dynamic programming problem

$$J_0(x) = \max_{u_t \in [0,1]} \left[\sum_{t=0}^{T-1} 2q^t \sqrt{u_t x_t} + pq^T \sqrt{x} \right], \qquad \text{where } x_{t+1} = x_t \cdot (1-u_t), \quad x_0 = x \geq 0.$$

This problem is not unlike 11–07. A look at that problem could be helpful.

(a) Consider first the finite-horizon problem with T a finite natural number. Show that its value function $J_t(x)$ can be written on the form $a_t q^t \sqrt{x}$ – whether or not p is positive, zero or negative.
(You need not find any difference equation for a_t to do this!)

(b) Let $T = +\infty$ and $p = 0$.

- State the Bellman equation for the value function.
- Show that it is satisfied by a function $A\sqrt{x}$ for some $A > 0$.