University of Oslo
Department of Economics

ECON5150 Mathematics 4

Monday 14 December 2009, 14:30–17:30

There are 3 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Give reasons for all your answers.

Grades given run from A (best) to E for passes, and F for fail.

Problem 1

Consider the dynamic programming problem

$$\max \ \mathsf{E}\Big[\sum_{t=0}^{T-1}\Big(\frac{1}{2}\Big)^t\big(\sqrt{u_t}+u_t\big)+\Big(\frac{1}{2}\Big)^T\big(\sqrt{x_T}+x_T\big)\Big]$$

subject to

$$x_{t+1}=(x_t-u_t)V_{t+1}, \quad x_0>0.$$

Here $V_t \in \{0,4\}$, all V_t are i.i.d., T is fixed, $u_t>0$ is the control, $\mathsf{Pr}[V_t=4]=1/2$, $\mathsf{Pr}[V_t=0]=1/2$. Let $K=\mathsf{E}[\ln(V_t)]$.

(a) Solve the above problem. (Assume that we get $x_t>0$ and $x_t-u_t>0$ all the time. Check this at the end.) (*Hint:* The formula for $J_t(x)$ does not change with t, except for one time-dependent constant.)

(b) Let $T=\infty$. Write out the Bellman equation. Try to solve the problem in this case. (*Hint:* For $J(x)$, make a guess, by using the results in part (a).)

(c) What do you need to show in order to know that you have found an optimal solution in part (b)?

Problem 2

Let C be the union of $(-\infty,0]$ and the intervals $\left[\dfrac{1}{2^{2k}},\dfrac{1}{2^{2k-1}}\right]$, $k=1,\ 2,\ \ldots$. Show that C is closed.

(Cont.)

Problem 3

Let X_t be a Markov chain random walk on $\{0, 1, \dots, N-1, N\}$, with transition probabilities p_{ij} defined by

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1-p \quad (i = 1, \dots, N-1); \quad \text{states 0 and } N \text{ are absorbing.}$$

The Markov chain X can be thought of as a gamble with bet 1 and $\mathsf{Pr}[\text{win}] = p = 1 - \mathsf{Pr}[\text{loss}]$, repeated until the player has 0 or N. Throughout this problem, we will assume $0 < p < 1$.

Define $T = \min\{t \geq 0;\ X_t \in \{0, N\}\}$, the first time of absorption.

(a) Let $\eta_i = \mathsf{Pr}[X_T = N | X_0 = i]$ and $\theta_i = \mathsf{E}[T | X_0 = i]$. Use first-step analysis to show that for $i = 1, \dots, N-1$, both η_i and θ_i must satisfy linear second-order difference equations of the form

$$pv_{i+1} - v_i + (1-p)v_{i-1} = K$$

and determine the constants $K = K_\eta$ for the equation for η, and $K = K_\theta$ for the equation for θ.

(b) Find the general solution of the difference equation for v, and find the particular solutions $\eta = (\eta_0, \eta_1, \dots, \eta_N)$ with $K = K_\eta$ and $\theta = (\theta_0, \theta_1, \dots, \theta_N)$ with $K = K_\theta$.

We now modify the gamble in two ways:

First, at state i, the player bets b_i, chosen as large as possible subject to the constraints $b_i \leq i$ (keeping the Markov chain positive) and $b_i \leq N - i$ (preventing the Markov chain from exceeding N; with this $b_i = \min\{i,\ N-i\}$, we either have $i - b_i = 0$ or $i + b_i = N$ (or both).

Second, when a player hits 0 or N, (s)he will exit the gambling house and a new player will enter at state 1.

The resulting Markov chain Z_t then has transition probabilities $\tilde{p}_{ij}$ defined by

$$\tilde{p}_{i,i+b_i} = p, \qquad \tilde{p}_{i,i-b_i} = 1-p \qquad (i = 1, \dots, N-1); \qquad \tilde{p}_{0,1} = \tilde{p}_{N,1} = 1$$

so that for the cases $N = 5$ and $N = 6$ the transition matrix $\widetilde{\mathbf{P}}$ becomes

(Cont.)

$$
\begin{array}{c} \\ 0\\1\\2\\3\\4\\5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array}\\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0\\
1-p & 0 & p & 0 & 0 & 0\\
1-p & 0 & 0 & 0 & p & 0\\
0 & 1-p & 0 & 0 & 0 & p\\
0 & 0 & 0 & 1-p & 0 & p\\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
\quad\text{and}\quad
\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
1-p & 0 & p & 0 & 0 & 0 & 0\\
1-p & 0 & 0 & 0 & p & 0 & 0\\
1-p & 0 & 0 & 0 & 0 & 0 & p\\
0 & 0 & 1-p & 0 & 0 & 0 & p\\
0 & 0 & 0 & 0 & 1-p & 0 & p\\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array},
$$

respectively.

(c) For each case $N = 5$ and $N = 6$, find the communication classes for Z, and decide for each class whether it is recurrent or transient.

(Specify at the following level of detail: For X, we have the accessibility relations $1 \to 2 \to \cdots \to N-2 \to N-1 \to N-2 \to \cdots \to 2 \to 1$, so that $\{1, 2, \dots, N-1\}$ all communicate; furthermore 0 only communicates with itself, and so does N; this gives three classes.)

(d) Let $N = 5$, and define $R = \min\{t \geq 1;\ Z_t = 1\}$. Find the distribution of R, conditional on $Z_0 = 1$ (i.e. find the numbers $\rho_n = \mathsf{Pr}[R = n | Z_0 = 1]$ for all positive integers n) and the conditional expectation $\mathsf{E}[R | Z_0 = 1]$. (Conditional on $Z_0 = 1$, R is the first return time to 1.)

(e) Let $N = 5$ still. Find the long-term mean occupation time m_1 of state 1 for Z, i.e.

$$m_1 = \lim_{t\to\infty} \Big(\frac{1}{t} \sum_{s=0}^{t-1} 1_{Z_s = 1} \Big).$$