Lecture 2

The "maximal number of linearly independent vectors in S":

k if there exists a linearly independent set of k vectors from S, but no such set of $k+1$.
($k = 0$ if $S = \{\vec{0}\}$)

Def. The rank $r(\vec{A})$ of a matrix $\vec{A}$, is the maximal number of linearly independent column vectors of $\vec{A}$.

Fact: equals the max # of lin. indep row vectors of $\vec{A}$

and

equals the "order" of the largest nonzero minor of $\vec{A}$

k if the minor is $k \times k$

(with $r(\vec{0}) = 0$.)

The first fact follows from the second, which indicates how to calculate.

An $m \times n$ matrix has rank $\leq \min\{m, n\}$.

If the rank <u>equals</u> $\min\{m, n\}$: "full rank".

(Otherwise: "rank-deficient".)

(If you ever see e.g. "full row rank", it means $r(\vec{A}) = m \leq n$)

Example 1: $\vec{A} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ has rank equal to <u>2</u>.

< 3 because the only 3×3 minor (namely $|\vec{A}|$) is zero

≥ 2 because $\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \neq 0$. (for "$\geq 2$" it suffices that one 2×2 minor is $\neq 0$)

Example 2:

$$\vec{A} = \begin{pmatrix} 11 & 21 & 31 & \cdots & 91 \\ 12 & 22 & 32 & \cdots & 92 \\ 13 & 23 & 33 & \cdots & 93 \\ 14 & 24 & 34 & \cdots & 94 \end{pmatrix}$$

has rank two as well.

Err... you don't want to calculate all 4×4 and all 3×3 minors to find out...?

Example 2 cont'd: how to calculate (by hand)?

Use elementary row or column operations!

Recall:
$$\begin{array}{c} \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} \\ -1 \end{array} = \begin{array}{c} \begin{vmatrix} 1 & 3 & 6 \\ 2 & 3 & 6 \\ 3 & 3 & 6 \end{vmatrix} \\ -2 \end{array} = 0.$$

Do this on the big rectangular matrix:

$$r\begin{array}{c}\begin{pmatrix} 11 & 21 & \cdots & 91 \\ 12 & 22 & & 92 \\ 13 & 23 & & 93 \\ 14 & 24 & \cdots & 94 \end{pmatrix} \\ -1 \end{array} = r\begin{pmatrix} 11 & 10 & \overbrace{20 \quad \cdots \quad 80}^{\text{can be deleted...}} \\ 12 & 10 & 20 \qquad\quad 80 \\ 13 & 10 & 20 \qquad\quad 80 \\ 14 & 10 & 20 \quad \cdots \quad 80 \end{pmatrix}$$

... because? -2, -3, $\vdots$, -8

$$= r\begin{array}{c} \qquad\quad \text{why?} \\ \begin{pmatrix} 11 & 1 \\ 12 & 1 \\ 13 & 1 \\ 14 & 1 \end{pmatrix} \end{array} = \begin{cases} \leq 2, & \text{since 2 columns} \\ \geq 2, & \text{since } \begin{vmatrix} 11 & 1 \\ 12 & 1 \end{vmatrix} \neq 0. \end{cases}$$

This approach is also useful for questions like: "Is $S = \{ \ldots \}$ linearly dependent?"

Example 3: [...]

Exam Q could be: Let $\vec{A}_t = ($[depends on t]$)$.
Find the rank, for each $t \in \mathbb{R}$.

Rank and linear eq. systems $\vec{A}\,\vec{X} = \vec{B}$

(Important!) fact: we have the equivalence

I

$$\vec{A}\,\vec{X} = \vec{B} \text{ has solution}$$
$$\text{if and only if}$$
$$r(\vec{A}) = r(\underbrace{\vec{A} \,\vdots\, \vec{B}}_{\text{the augmented coeff. matrix.}})$$

In II and III, assume $\vec{B} = \vec{b}$ is a vector.

II

Assume $r(\vec{A}) = r(\vec{A} \,\vdots\, \vec{b}) = r$
so a solution exists.
Let $\vec{A}$ be $m \times n$. Then

* there are $n - r$ degrees of freedom
* there are $m - r$ superfluous eq's.

But there is more. Assume we do have solution, $r(\vec{A}) = r(\vec{A} \,\vdots\, \vec{b}) = r$.

III

- Let an $r \times r$ submatrix $\vec{M}$ of $\vec{A}$ ← not of $(\vec{A} \,\vdots\, \vec{b})$! have $|\vec{M}| \neq 0$.
  (that is, $|\vec{M}|$ is one $r \times r$ minor).
- Mark off the elements of $\vec{M}$ in $\vec{A}$. Then:
  → the "other" rows (not intersecting $\vec{M}$) can be deleted
  → the "other" col's correspond to x_i we can choose freely.

Why (I)? You don't need proof, but you should catch what goes on:

- $\vec{A}\vec{x} = x_1 \vec{a}^{(1)} + \ldots + x_n a^{(n)}$ ← columns of $\vec{A}$

- If some $a^{(i)}$ can be written as linear combination of the others, then do that [eliminating a degree of freedom, if there is solution!]

  Repeat until r linearly indep. col's remain.

- If $\sum x_i \vec{a}^{(i)} = \vec{b}$ then $\vec{b}$ can be written as a linear combination.

  [Fancy math speak: $\vec{b} \in$ the span of the col's]

So augmenting with $\vec{b}$ cannot increase the # of lin. indep. vectors if there is a sol'n; conversely, if it does not do so, we do have a solution.

- Solution ⇔ $\vec{A}$ and $(\vec{A} : \vec{b})$ have same # of lin. indep. col's ⇔ same rank.

- For $\vec{A}\overleftrightarrow{X} = \vec{B}$, this must hold for $\vec{A}$ vs $(\vec{A} : \vec{b}_i)$, every column $\vec{b}_i$ of $\vec{B}$.

Why II and III ?

Loosely:

- $\#$ of lin. indep rows $= \#$ lin. indep col's.
- rows – i.e. eq's – that can be written in terms of the others, can be deleted.
- An $r \times r$ minor has the largest possible number of "lin. indep left-hand sides" and thus also "right-hand sides" since we assume we have solution!

Delete these superfluous eq's.

- Left with r eq's in n unknowns.
- If after "moving $n-r$ var's to the RHS" we have an invertible eq. system, then no matter how we choose those $n-r$, we have unique values for the rest.

Picture: If $r(\vec{A}) = r(\vec{A} : \vec{b})$: and, e.g

$$\vec{A} = \begin{pmatrix} \cdot & \odot & \cdot & \cdot & \odot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \odot & \cdot & \cdot & \odot & \cdot \end{pmatrix}$$

where the circled elements form a largest nonzero minor, then we have 4 deg's of freedom and

- we <u>can</u> delete eq $\#2$
- we <u>can</u> choose x_1, x_3, x_4, x_6 freely.

Example: Consider
$$\begin{pmatrix} 1 & 6 & 2 & 1 \\ -2 & -5 & 0 & -1 \\ 3 & 4 & p & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ q \\ 2 \end{pmatrix}$$

For each p, q, decide the number of solutions / degrees of freedom. [Can there possibly be unique sol'n?]

- $r(\vec{A}) \geq 2$ (e.g., $\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} \neq 0$.)
- $\vec{a}^{(4)} = \frac{1}{7}\left(\vec{a}^{(1)} + \vec{a}^{(2)}\right)$ so

$$r(\vec{A}) = r\underbrace{\begin{pmatrix} 1 & 6 & 2 \\ -2 & -5 & 0 \\ 3 & 4 & p \end{pmatrix}}_{\text{has determinant } 7p+14} = \begin{cases} 3 & \text{for } p \neq -2 \\ 2 & \text{for } p = -2 \end{cases}$$

Solution with 1 d.f. (x_4 can be free) for $p \neq -2$.

Let $p = -2$ $\quad r(\vec{A} \mid \vec{b}) = r(\vec{a}^{(1)} \mid \vec{a}^{(2)} \mid \vec{b})$ (WHY?)

$$= r\begin{pmatrix} 1 & 6 & 1 \\ -2 & -5 & q \\ 3 & 4 & 2 \end{pmatrix} \begin{matrix} \\ +1 \\ \\ \end{matrix} \; \begin{matrix} \cdot 2 \\ \\ \\ \end{matrix} = r\begin{pmatrix} 1 & 6 & 1 \\ -2 & -5 & q \\ 0 & -13 & q \end{pmatrix}$$

determinant $7q + 13(q+2) = 20q + 26$.

So for $p = -2,\ q \neq -\frac{13}{10}$: <u>no solution</u>

For $p = -2,\ q = -\frac{13}{10}$: $r(\vec{A}) = r(\vec{A} \mid \vec{b}) = 2$

and solution with two degrees of freedom (e.g. x_3 and x_4)

Example:

Does
$$\begin{pmatrix} 9 & 41 \\ 8 & 42 \\ 7 & 43 \\ \vdots & \vdots \\ 1 & 49 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$
have

no, one or $\underbrace{\text{infinitely}}$ many solutions?

if so: only one degree of freedom! (WHY?)

Non-prop col's, so $r(\vec{A}) = 2$. (what can we say already?)

$$r(\vec{A} \,\vdots\, \vec{b}) = r \begin{pmatrix} 9 & 41 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 49 & 1 \end{pmatrix} = 2$$

+1 second col becomes 50 · third

So the eq. system is $\Leftrightarrow$ to

$$\begin{pmatrix} 9 & 41 \\ 8 & 42 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$, unique sol'n.

Eigenvalues & eigenvectors

[This is both an exam topic in its own right, and something used for other topics!]

The big Q: When is $\vec{A}\vec{x} = \lambda \vec{x}$ ⊛
for some $\vec{x} \neq \vec{0}$ and some number λ?

Def. Fix a matrix $\vec{A}$, (necessarily square).
If for some $\vec{x} \neq \vec{0}$ there exists a number λ such that $\vec{A}\vec{x} = \lambda\vec{x}$, we say that
$\vec{x}$ is an eigenvector of $\vec{A}$ with corresponding eigenvalue λ

Why bother?

Ex 1: We shall cover the following facts in this course: Let $f \in C^2(\mathbb{R}^n)$. We have

the Hessian has all eigenvalues positive everywhere
$\Rightarrow$ f strictly convex $\Rightarrow$ f convex
$\Rightarrow$ the Hessian has all eigenvalues nonnegative everywhere.

Ex. 2 Let $\vec{x} = \vec{x}(t)$ follow the differential eq.

constant matrix.

$$\frac{d}{dt}\vec{x}(t) = \vec{A}\,\vec{x}(t)$$

If some eigenvalue of $\vec{A}$ is >0, then some particular solutions diverge (instability)

Ex. 3: See this link to BI's "Math 2" course, pp 1.-8.

http://www.dr-eriksen.no/teaching/GRA6035/2010/lecture3-hand.pdf

(We will <u>not</u> cover his slides 17-18, and barely touch his pp 20 ff.)

Back on track:

$$\vec{A}\,\vec{x} = \lambda\,\vec{x}.$$

Example: Show that $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvalue for $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

Solution: $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -7 \\ 21 \end{pmatrix} = -7\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

OK, with $\lambda = -7$.

Verifying that was easy.

Example: Show that $\mu = 3$ is an eigenvalue of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

We need to show that there exists $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}.$$

We could: Rewrite as
$$\begin{aligned} 2x + 3y - 3x &= 0 \quad (e) \\ 3x - 6y - 3y &= 0 \end{aligned}$$
and solve. That would give us $\begin{pmatrix} x \\ y \end{pmatrix}$ (one degree of freedom)

But as we were not asked for $\begin{pmatrix} x \\ y \end{pmatrix}$, only to show some $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ exists: Eq. (e) has non-invertible coeff. matrix!

$\begin{vmatrix} -1 & 3 \\ 3 & -9 \end{vmatrix} = 0$. Non-unique sol'n.

So how to find from scratch?

Note: $\vec{A}\vec{x} = \lambda\vec{x}$

$$\Leftrightarrow \quad (\vec{A} - \lambda\vec{I})\vec{x} = \vec{0}.$$

We want those λ for which there is a nonzero – i.e, non-unique! – solution.

That is: When $\underline{\det(\vec{A} - \lambda\vec{I}) = 0}$ ⓒ

ⓒ is called the <u>characteristic equation</u> of $\vec{A}$.

$p(\lambda) := \det(\vec{A} - \lambda\vec{I})$ is called the <u>characteristic polynomial</u>. It is an n^{th} order polynomial when $\vec{A}$ is $n \times n$. Leading coefficient: $(-1)^n$.

So, "method" for finding eigenvalues:

- Calculate $p(\lambda) = \det(\vec{A} - \lambda\vec{I})$
- Solve $p(\lambda) = 0$ for λ.

But ... if $n > 2$ ...? Or even > 4?

<u>If</u> we have found such a λ, its associated eigenvector is found by:

solving $(\vec{A} - \lambda\vec{I})\vec{x} = \vec{0}$ for $\vec{x}$, with this λ here.